PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 2, 1974

PERTURBATIONS OF TYPE I VON NEUMANN ALGEBRAS

JOHN PHILLIPS

In a recent paper, R. V. Kadison and D. Kastler studied a certain metric on the family of von Neumann algebras defined on a fixed Hilbert space. The distance between two von Neumann algebras was defined to be the Hausdorff distance between their unit balls. They showed that if two von Neumann algebras were sufficiently close, then their central portions of type $K (K = I, I_n, II, II_1, II_\infty, III)$ were also close.

In the introduction to their paper, they conjectured that neighbouring von Neumann algebras must actually be unitarily equivalent. It is the purpose of this paper to prove this conjecture in the case that one of the algebras is of type I. The question of "inner" equivalence is left open. (Can the unitary equivalence be implemented by a unitary operator in the von Neumann algebra generated by the two neighbouring algebras?)

2. Notation and definitions. If $\mathscr{A}$ is a set of bounded linear operators on the Hilbert space $\mathscr{H}$, then we use $\mathscr{A}'$ to denote the set of all bounded linear operators on $\mathscr{H}$ which commute with every element of $\mathscr{A}$. We use the notation $\mathscr{A}_1$ to denote the set of all operators in $\mathscr{A}$ whose bound is less than or equal to 1. The algebra of all bounded operators on $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$. If $\mathscr{A}$ is an algebra of operators on $\mathscr{H}$ with identity, then the identity of $\mathscr{A}$ is denoted by $1_{\mathscr{A}}$. However, the identity operator on $\mathscr{H}$ will be denoted by 1. For each subset $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$ and bounded operator A on $\mathscr{H}$ we let

$$\|A - \mathscr{F}\| = \inf\{\|A - F\|: F \text{ in } \mathscr{F}\} .$$

DEFINITION 2.1. If $\mathscr{A}$ and $\mathscr{B}$ are linear subspaces of $\mathscr{B}(\mathscr{H})$,

$$\|\mathscr{A} - \mathscr{B}\| = \sup\{\|A - \mathscr{B}_1\|, \|B - \mathscr{A}_1\|: A \text{ in } \mathscr{A}_1, B \text{ in} \mathscr{B}_1\} .$$

If $\mathscr{S}$ is a subset of $\mathscr{B}(\mathscr{H})$, then the closure of $\mathscr{S}$ in the ultraweak topology is denoted by $\mathscr{S}^-$. Also, co $\mathscr{S}$ will denote the set of convex linear combinations of elements of $\mathscr{S}$.

As in [4], we make frequent use of the monotone increasing function α: $[0, 1/8] \rightarrow [0, 5/8]$ defined by $\alpha(a) = a + 1/2 - (1/4 - 2a)^{1/2}$. We also make use of the following estimates for α: