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A LOCAL ESTIMATE FOR TYPICALLY REAL FUNCTIONS

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In this paper it is shown that for each typically real function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ the local estimate $n - a_n \leqq (1/6)n(n^2 - 1)(2 - a_2)$ holds, $n = 2, 3, \cdots$. The constant $(1/6)n(n^2 - 1)$ is best possible.

Bombieri [1] proved the existence of constants γ_n such that for each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the unit disk D,

$$|\operatorname{Re}(n - a_n)| \leqq \gamma_n \operatorname{Re}(2 - a_2), \qquad n = 2, 3, \cdots. \tag{1}$$

Hummel [2] showed that if in addition f maps D onto a domain starlike with respect to the origin, then $|n - a_n| \leqq \gamma_n |2 - a_2|$ for the value

$$\gamma_n = n(n^2 - 1)/6 ; \tag{2}$$

furthermore, this choice of γ_n is best possible. In this paper we shall show that (2) is also the best possible constant in (1) for the collection of univalent functions with real coefficients. More generally, we answer this question for the set T of typically real functions.

DEFINITION 1. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in D is said to be *typically real* provided $f(z)$ is real if and only if z is real.

The class T was introduced by Rogosinski [5], [6]. Among other things he showed that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T$, then a_n is real and $|a_n| \leqq n$, $n = 2, 3, \cdots$. Note that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in D and has real coefficients, then $f(\bar{z}) = \overline{f(z)}$. From this fact it easily follows that $f \in T$.

We now introduce a family of polynomials $P_n(t)$ closely related to the Chebyshev polynomials of the second kind.

DEFINITION 2. For each n, $n = 1, 2, \cdots$, set

$$r = \left[\frac{n-1}{2}\right], \quad P_n(t) = \sum_{k=0}^{r} (-1)^k \binom{n-k-1}{k} t^{n-2k-1},$$

where t is real.

DEFINITION 3. Let c_n be the largest critical point of $P_n(t)$, $n = 3, 4, \cdots$.