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BIHOLOMORPHIC APPROXIMATION OF PLANAR DOMAINS

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This paper establishes the existence of a domain (open connected subset) B of the complex plane C such that for every domain $\Omega \subset C$ and every compact set $K \subset \Omega$, there is a biholomorphic embedding $e: B \to \Omega$, such that $K \subset e(B) \subset \mathrm{cl}\,[e(B)] \subset \Omega$.

1. Introduction. Let Ω_1 and Ω_2 be domains (i.e., open connected sets) in the complex plane C such that $\mathrm{cl}\,\Omega_1 \subset \Omega_2$ (cl = closure). A domain Ω is a biholomorphic approximation of Ω_1 with respect to Ω_2 provided that there exists an invertible holomorphic function e defined on Ω such that

$$\mathrm{cl}\,\Omega_1 \subset e(\Omega) \subset \mathrm{cl}\,[e(\Omega)] \subset \Omega_2 .$$

The mapping e is a biholomorphic embedding (bh-embedding) of Ω into Ω_2. (Ω may also be considered a biholomorphic approximation of Ω_2 with respect to Ω_1.)

Homeomorphic domains may, of course, be biholomorphically inequivalent, and, moreover, may not even be close biholomorphic approximations of each other. For example, let $A(r, s) = \{z \in C: r < |z| < s\}$ when $0 < r < s < \infty$. Suppose that $0 < \varepsilon < 1 < t < \infty$ and that e is a bh-embedding of $A = A(r, s)$ such that

$$\mathrm{cl}\,A(1, t) \subset e(A) \subset \mathrm{cl}\,[e(A)] \subset A(1 - \varepsilon, t + \varepsilon) .$$

By taking the modules of these ring domains (cf. [1]) we obtain the inequality $t < s/r < (t + \varepsilon)/(1 - \varepsilon)$ which is precisely the condition r and s must satisfy for such an embedding e to exist.

Our main result establishes the existence of a domain $B \subset C$ which is a biholomorphic approximation of every bounded domain Ω_1 with respect to every domain Ω_2 containing $\mathrm{cl}\,\Omega_1$.

2. The main theorem. Let $\hat{C}$ denote the Riemann sphere.

THEOREM 2.1. *There exists a domain $B \subset C$ such that for every domain $\Omega \subset \hat{C}$ and for every compact set $K \subset \Omega$ other than $\hat{C}$ there exists a biholomorphic embedding $e: B \to \Omega$ such that $K \subset e(B) \subset \mathrm{cl}\,[e(B)] \subset \Omega$.*

REMARK. Actually such an embedding will exist if Ω is any connected Riemann surface (without boundary) and $K \subset \Omega$ is any planar compact surface other than $\hat{C}$. ("Planar" means homeomorphic