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COMPLETELY DECOMPOSABLE GROUPS WHICH ADMIT ONLY NILPOTENT MULTIPLICATIONS

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A triangle of size n is a collection $\{A_u\}$ of $n(n+1)/2$ (not necessarily distinct) rank one torsion-free abelian groups indexed by all integer sequences of the form $u = i, i+1, \cdots, i+j$ with $1 \leqq i \leqq i+j \leqq n$, satisfying $T(A_u) + T(A_s) \leqq T(A_{us})$ for all consecutive sequences u, s. Here $T(A_v)$ denotes the type of the rank one torsion-free abelian group A_v. If $A = \bigoplus_{i \in I} A_i$ is a direct sum of rank one torsion-free abelian groups A_i, let $\Delta(A) = \sup\{n \mid \exists$ a triangle of size n of groups chosen, possibly with repetitions, from $\{A_i \mid i \in I\}\}$, $\Delta'(A) = \sup\{n \mid \exists$ a triangle of size n of groups chosen without repetition from $\{A_i \mid i \in I\}\}$. An abelian group $(G, +)$ is radical iff whenever $(R, +, \cdot)$ is a ring with $(R, +) \cong (G, +)$ there exists a positive integer n with $R^n = (0)$.

THEOREM. *Let $A = \bigoplus_{i \in I} A_i$ be such that $\{T(A_i) \mid i \in I\}$ is an ordered set and $\Delta(A) < \infty$. Then A is radical.*

THEOREM. *Let $A = \bigoplus_{i \in I} A_i$ be such that $\Delta'(A) = \infty$. Then A is not radical.*

THEOREM. *Let $A = \bigoplus_{i \in I} A_i$, $B = \bigoplus_{j \in J} B_j$ be such that $\Delta(A) < \infty$, $\Delta(B) < \infty$. Then if $\{T(A_i) \mid i \in I\} \cup \{T(B_j) \mid j \in J\}$ is an ordered set $A \oplus B$ is radical. A bound is given for the index of nilpotency of any multiplication on $A \oplus B$.*

1. Preliminaries. Several authors ([2], [3], [4], [5]) have studied the class of abelian groups $(A, +)$ which admit only a trivial ring structure; i.e., if $(R, +, \cdot)$ is a ring with $(R, +) \cong (A, +)$, then $R^2 = (0)$. These are called nil groups.

In [6] a larger class was introduced—abelian groups which admit only nilpotent multiplications. More precisely:

DEFINITION 1.1. An abelian group $(A, +)$ is a radical group iff whenever $(R, +, \cdot)$ is a ring with $(R, +) \cong (A, +)$, we have $R^n = 0$ for some positive integer n.

In [6], using the techniques of [1], the class of finite rank torsion free radical groups was studied, and it was shown that this class is closed under finite direct sums.

In this paper we study completely decomposable radical groups. We work toward the goals of characterizing such groups and of obtaining information on finite direct sums of such groups.

Throughout, the word "group" means torsion-free abelian group. We let $A = \bigoplus_{i \in I} A_i$, $B = \bigoplus_{j \in J} B_j$ be arbitrary completely decompo-