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GENERALIZED LERCH ZETA FUNCTION

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The purpose of this paper is to establish certain properties of the generalized Lerch zeta function $\theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^{(n+a)^b}$. The main result yields another infinite series representation for θ. A generalization of Hardy's relation follows as an immediate corollary.

1. Introduction. The function $\Phi(z, \nu, a)$ defined by the power series

$$(1) \qquad \Phi(z, \nu, a) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^n ,$$

for $|z| < 1$, $0 < a \leqq 1$ and arbitrary ν, is called Lerch's zeta function. For $z = 1$, this function becomes Hurwitz' zeta function

$$(2) \qquad \Phi(1, \nu, a) = \zeta(\nu, a) = \sum_{n=0}^{\infty} (n + a)^{-\nu}, \ \mathrm{Re}\, \nu > 1 \text{ and } 0 < a \leqq 1 .$$

Lerch's function has been extensively investigated in [1], [2], [3], [5, v. 1, p. 27], [7], [8], and [12]. One important result

$$(3) \qquad \Phi(z, \nu, a) = \Gamma(1 - \nu) z^{-a} (\log 1/z)^{\nu-1} + z^{-a} \sum_{r=0}^{\infty} \zeta(\nu - r, a) \frac{(\log z)^r}{r!} ,$$

for $|\log z| < 2\pi$, $0 < a \leqq 1$, $\nu \neq 1, 2, 3, \cdots$, which transforms Lerch's series into another series, is derived in Erdélyi [5, v. 1, p. 29] by using Lerch's transformation formula and Hurwitz' series for the Hurwitz zeta function. Hardy's relation (see Hardy [7] and Mellin [10]) follows immediately from (3):

$$(4) \qquad \lim_{z \to 1} \{\Phi(z, \nu, a) - \Gamma(1 - \nu) (\log 1/z)^{\nu-1} z^{-a}\} = \zeta(\nu, a) .$$

The purpose of this paper is to establish certain properties of the function $\theta(z, \nu, a, b)$ where

$$(5) \qquad \theta(z, \nu, a, b) = \sum_{n=0}^{\infty} (n + a)^{-\nu} z^{(n+a)^b}, \text{ for } |z| < 1, 0 < a \leqq 1, 0 < b .$$

It is appropriate to call θ the generalized Lerch zeta function because

$$\theta(z, \nu, a, 1) = z^a \Phi(z, \nu, a) .$$

Using an approach which is more direct than the above mentioned derivation of equation (3), we will establish