PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

DECOMPOSITION THEOREMS FOR 3-CONVEX SUBSETS OF THE PLANE

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Let S be a 3-convex subset of the plane. If $(\mathrm{cl}\, S \sim S) \subseteq \mathrm{int}\,(\mathrm{cl}\, S)$ or if $(\mathrm{cl}\, S \sim S) \subseteq \mathrm{bdry}\,(\mathrm{cl}\, S)$, then S is expressible as a union of four or fewer convex sets. Otherwise, S is a union of six or fewer convex sets. In each case, the bound is best possible.

1. Introduction. Let S be a subset of R^d. Then S is said to be 3-*convex* iff for every three distinct points in S, at least one of the segments determined by these points lies in S. Valentine [2] has proved that for S a closed, 3-convex subset of the plane, S is expressible as a union of three or fewer closed convex sets. We are interested in obtaining a similar decomposition without requiring the set S to be closed. The following definitions and results obtained by Valentine will be useful.

For $S \subseteq R^d$, a point x in S is a *point of local convexity of* S iff there is some neighborhood U of x such that, if $y, z \in S \cap U$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Let S be a closed, connected, 3-convex subset of the plane, and let Q denote the closure of the set of isolated lnc points of S. Valentine has proved that for S not convex, then $\mathrm{card}\, Q \geqq 1$, Q lies in the convex kernel of S, and $Q \subseteq \mathrm{bdry}\,(\mathrm{conv}\, Q)$. An *edge* of $\mathrm{bdry}\,(\mathrm{conv}\, Q)$ is a closed segment (or ray) in $\mathrm{bdry}\,(\mathrm{conv}\, Q)$ whose endpoints are in Q. We define a *leaf* of S in the following manner: In case $\mathrm{card}\, Q \geqq 3$, let L be the line determined by an edge of $\mathrm{bdry}\,(\mathrm{conv}\, Q)$, L_1, L_2 the corresponding open halfspaces. Then L supports $\mathrm{conv}\, Q$, and we may assume $\mathrm{conv}\, Q \subseteq \mathrm{cl}\,(L_1)$. We define $W = \mathrm{cl}\,(L_2 \cap S)$ to be a *leaf* of S. For $2 \geqq \mathrm{card}\, Q \geqq 1$, constructions used by Valentine may be employed to decompose S into two closed convex sets, and we define each of these convex sets to be a *leaf* of S.

By Valentine's results, every point of S is either in $\mathrm{conv}\, Q$ or in some leaf W of S (or both), and every leaf W is convex. Moreover, Valentine obtains his decomposition of S by showing that for any collection $\{s_i\}$ of disjoint edges of $\mathrm{bdry}\,(\mathrm{conv}\, Q)$, with $\{W_i\}$ the corresponding collection of leaves, $\mathrm{conv}\, Q \cup (\bigcup W_i)$ is closed and convex.

Finally, we will use the following familiar definitions: For x, y in S, we say x *see* y *via* S iff the corresponding segment $[x, y]$ lies in S. A subset T of S is *visually independent via* S iff for every