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DIRECT SUM SUBSET DECOMPOSITIONS OF Z

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Let Z be the set of integers. In this paper it is shown that there is no effective characterization of all direct sum subset decompositions of Z i.e., where $A+B=Z$ and the sums are distinct. Further the result is generalized to include decompositions of a product of sets where Z is a set in the product, and to cases where the number of subsets in the decomposition is greater than two.

The question of characterizing all direct sum subset decompositions for Z, the infinite cyclic group, seems first to have been raised explicitly by de Bruijn [1]. It was mentioned again by de Bruijn [2] in 1956, and Long [5] in 1967. The notation $A \oplus B$ will denote $A + B$ where the sums are distinct. Without loss of generality we will assume 0 is a member of each summand.

For the semi-group Z^+ there is a particularly nice characterization of all direct sum decompositions. The result, which was implicit from the work of de Bruijn [2], was first explicitly by Long [5]. It is the following:

THEOREM 1. *Let* $|A| = |B| = \infty$. $A \oplus B = Z^+$ *if and only if there exists an infinite sequence of integers* $\{m_i\}_{i\geqq 1}$ *with* $m_i \geqq 2$ *for all* i, *such that* A *and* B *are the sets of all finite sums of the form*

$$a = \sum x_{2i} M_{2i}$$
$$b = \sum x_{2i+1} M_{2i+1}$$

respectively, where $0 \leqq x_i < m_{i+1}$ *for* $i \geqq 0$ *where* $M_0 = 1$ *and* $M_i = \prod_{j=1}^{i} m_j$ *for* $i \geqq 1$.

In case $|A| < \infty$ or $|B| < \infty$, a similar characterization holds with the change that the sequence $\{m_i\}$ will be of finite length r and the only restriction on x_r is that it be nonnegative.

A distinguishing characteristic of decompositions obtained as in Theorem 1 is that either A or B has the property that each of its elements is a multiple of some integer $m \geqq 2$ and it has been conjectured that this property would necessarily hold for any decomposition of Z. The following theorem shows that this is not the case and that the decomposing sets A and B can be quite arbitrary. It follows that there is no real possibility of effectively characterizing A and B. We do obtain a rather weak characterization in Theorem 3.