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THE NORM OF A CERTAIN DERIVATION

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J. C. Stampfli has asked whether the norm of the derivation $\mathfrak{D}_T: A \to TA - AT$ as a mapping of the subalgebra $\mathfrak{A}$ of $\mathfrak{B}(H)$ into $\mathfrak{B}(H)$ is given by $\inf\{2\|T - A'\|: A' \in \mathfrak{A}'\}$. That this need not be the case is shown through an example in 4×4 matrices.

H is a Hilbert space. $\mathfrak{B}(H)$ is the algebra of all bounded linear operators on H. $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}(H)$ and $\mathfrak{A}'$ is the commutant of $\mathfrak{A}$.

In [6], J. C. Stampfli proved that the norm of $\mathfrak{D}_T$ as a mapping of $\mathfrak{B}(H)$ into itself is precisely $2 \inf_\lambda \|T - \lambda\|$. Thus the question about $\|\mathfrak{D}_T\|$ as a mapping from $\mathfrak{A}$ to $\mathfrak{B}(H)$ naturally arises. In addition, Kadison, Lance, and Ringrose [2, Theorem 3.1] show that if $T = T^*$ and $\mathfrak{D}_T$ maps $\mathfrak{A}$ into itself, then $\|\mathfrak{D}_T\| = \inf\{2\|T - A'\|: A' \in \mathfrak{A}'\}$. Our example will have T self-adjoint, which shows that their hypothesis $\mathfrak{D}_T(\mathfrak{A}) \subset \mathfrak{A}$ is not inessential.

For our example, we take H to be complex four-dimensional Hilbert spaces; elements of H are to be thought of as column 4-vectors, and elements of $\mathfrak{B}(H)$ as 4×4 matrices. We take $\mathfrak{A}$ to be the subalgebra of diagonal matrices, so $\mathfrak{A}' = \mathfrak{A}$.

For T we take the Hermitian matrix

$$T = \frac{1}{12}\begin{pmatrix} 1 & -4 & \frac{1}{\sqrt{14}}(-5+6i\sqrt{3}) & \frac{1}{\sqrt{14}}(-5-6i\sqrt{3}) \\ -4 & 4 & -2\sqrt{14} & -2\sqrt{14} \\ \frac{1}{\sqrt{14}}(-5-6i\sqrt{3}) & -2\sqrt{14} & \frac{7}{2} & \frac{1}{14}(-95+12i\sqrt{3}) \\ \frac{1}{\sqrt{14}}(-5+6i\sqrt{3}) & -2\sqrt{14} & \frac{1}{14}(-95-12i\sqrt{3}) & \frac{7}{2} \end{pmatrix}$$

T is of the form $P - Q$ where P and Q are self-adjoint projections. The range of P is two-dimensional and is spanned by the orthogonal unit vectors

$$p^{(1)} = \frac{1}{2\sqrt{3}}\left(1, -1 + i\sqrt{3}, \frac{1}{\sqrt{14}}(4 - 5i\sqrt{3}), \frac{1}{\sqrt{14}}(-2 + i\sqrt{3})\right),$$

$$p^{(2)} = \frac{1}{2\sqrt{3}}\left(1, -1 - i\sqrt{3}, \frac{1}{\sqrt{14}}(-2 - i\sqrt{3}), \frac{1}{\sqrt{14}}(4 + 5i\sqrt{3})\right);$$

the range of Q is one-dimensional and is spanned by the unit vector