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# HOLLOW MODULES AND LOCAL ENDOMORPHISM RINGS

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**This is a study of the conditions under which smallness of proper submodules will influence the structure of the endomorphism ring of a module. The case in which that endomorphism ring becomes local is of special interest. Further facts about hollow modules are also established.**

1. Introduction. Throughout this paper, $R$ will denote an associative ring with unit and $R$-Mod the category of left $R$-modules. In [3], we studied elements of $R$-Mod such that every strictly decreasing sequence of submodules eventually became small. Such modules were said to have finite spanning dimension and we showed that each such could be represented as the sum of a unique number of hollow submodules. A module was said to be hollow if every submodule were small. That is, $X$ is hollow if $X$, $Y_1$, $Y_2 \in R$-Mod, $Y_1, Y_2 \subseteqq X$, and $Y_1 + Y_2 = X$ imply either $Y_1 = X$ or $Y_2 = X$. In this paper, we present some theorems about hollow modules and, in particular, we investigate the circumstances under which certain hollow modules have local endomorphism rings. By a local ring we simply mean a ring with a unique maximal left ideal and we assume neither chain conditions nor commutativity.

To a certain extent, hollow modules play the same role for modules with finite spanning dimension which simple modules play for modules with descending chain condition. For example, theorems about hollow modules having local endomorphism rings are extensions of Shur's lemma.

In § 2, we establish basic facts about hollow modules and in § 3 we introduce the idealizer and show its relation to endomorphism rings. In § 4, we present eight theorems which show that, under certain conditions, a hollow module has a local endomorphism ring. In § 5, we present some miscellaneous results on when a maximal left ideal is 2-sided and we show that the abelian group of $p$-adic integers is hollow.

2. Some preliminaries. We recall that if $X$ is an element of $R$-Mod and $Y_1$ is a submodule of $X$, then $Y_1$ is called small if $Y_1 + Y_2 = X$ for some submodule $Y_2$ of $X$ implies $Y_2 = X$. It is easy to show that the image of a small submodule is again small and we will have use for this fact later.