PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

ON LOEWY LENGTH OF RINGS

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Associated with each ring R over which every nonzero right module has a minimal submodule is an ordinal number called its right (lower) Loewy length. The concern here is with the various possible left and right Loewy lengths of such rings with zero radical and with the possible right-left symmetry of this minimal submodule condition. In particular, if R has finite right Loewy length n then R has left Loewy length $\leqq 2^n - 1$.

All rings are associative rings with identity. Denoting the socle of a module M by $\mathrm{Soc}(M)$, the (*lower*) *Loewy series* for M is defined transfinitely by: $S_0 = 0$, $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(M/S_\alpha)$ and, if α is a limit ordinal, $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. (See Bass [1, p. 470].) If $M = S_\alpha$ for some ordinal number α then M is called a (*lower*) *Loewy module*. The Loewy length of such a module is $L(M) = \gamma$, the least ordinal γ with $M = S_\gamma$. We call a ring R a *right* (resp., *left*) *Loewy ring* in case the regular representation R_R (resp., ${}_RR$) is a Loewy module. (Năstăsescu and Popescu [6] use the term "semi-artinian" to denote such a ring.) A ring R is easily seen to be a right Loewy ring if and only if each of its right modules has a nontrivial (hence, essential) socle. Over such a ring each right module M is a Loewy module of length $L(M) \leqq L(R_R)$ and, since R_R is finitely generated, $L(R_R)$ cannot be a limit ordinal.

As part of his Theorem P, Bass [1] proved that a ring R is left perfect (i.e., its (Jacobson) radical $J = J(R)$ is left T-nilpotent and R/J is semisimple) if and only if R is right Loewy and contains no infinite orthogonal set of idempotents. No doubt inspired by this result, Năstăsescu and Popescu [6] proved that a ring R is right Loewy if and only if its radical J is left T-nilpotent and R/J is right Loewy. Thus we are led to study Loewy rings with zero radical. After first modifying an example of Osofsky [7] to show that there are primitive left and right Loewy rings of arbitrary infinite left and right lengths and that a primitive right Loewy ring need not be left Loewy, we prove that a right Loewy ring of finite length must also be left Loewy of finite length, but that these two lengths are neither independent nor necessarily equal. Then, recalling that the left and right Loewy series for a (von Neumann) regular ring are the same, we show that there exist both commutative and primitive regular Loewy rings of arbitrary length.

1. Right Loewy length vs. left.

Osofsky [7], in answer to a