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SEMIPERFECT RINGS WITH ABELIAN ADJOINT GROUP

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A structure theorem is proved for semiperfect rings (possibly with no identity) which have an abelian adjoint group. This is used to give conditions when such a ring is finite or commutative. In particular, a semiperfect ring with identity is finite if its group of units is finitely generated and abelian. Additional information is obtained if the adjoint group is cyclic.

1. Preliminaries. Throughout this paper all rings will be associative but need *not* contain an identity element. If R is a ring, the *adjoint group* of R is the set R^0 of all elements of R which have inverses with respect to the operation $a \circ b = a + b - ab$. This operation will be called *adjoint multiplication*. If R has an identity the multiplicative group of units of R will be denoted by R^*. It is well known that R^* and R^0 are isomorphic groups. The additive group of a ring R will be denoted by R^+.

If R is a ring, a left R-module X will be called *G-unital* if $R^0X = 0$. If R has an identity this is equivalent to the condition that $ux = x$ for all $u \in R^*$ and $x \in X$, and so agrees with the usage of this term in [4]. A bimodule is G-unital if it is G-unital on both sides.

Let S and A be rings, let X be an $S - A$ bimodule and let Y be an $A - S$ bimodule. The *semidirect sum* $\begin{bmatrix} S & X \\ Y & A \end{bmatrix}$ is defined to be the set of all 2×2 "matrices" with components as shown. This is a ring if we use componentwise addition and multiplication

$$\begin{pmatrix} s & x \\ y & a \end{pmatrix}\begin{pmatrix} s' & x' \\ y' & a' \end{pmatrix} = \begin{pmatrix} ss' & sx' + xa' \\ ys' + ay' & aa' \end{pmatrix}.$$

The next proposition characterizes the adjoint group of such a ring and its routine proof is left to the reader.

PROPOSITION 1. *Let S and A be rings, let X be an $S - A$ bimodule and let Y be an $A - S$ bimodule. Then* $\begin{bmatrix} S & X \\ Y & A \end{bmatrix}^0 = \begin{bmatrix} S^0 & X \\ Y & A^0 \end{bmatrix}$ *and this group is abelian if and only if S^0 and A^0 are abelian and both X and Y are G-unital. Moreover, when this is the case the adjoint group* $\begin{bmatrix} S & X \\ Y & A \end{bmatrix}^0$ *is isomorphic to the direct product of the adjoint groups S^0 and A^0 and the additive groups X and Y.*

2. The structure theorem. The *Jacobson radical* of a ring R