PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 1, 1974

MAPPINGS BY PARALLEL NORMALS PRESERVING PRINCIPAL DIRECTIONS

DIMITRI KOUTROUFIOTIS

Two smooth surfaces $S, \bar{S}$ with positive Gaussian curvature and with the same closed hemisphere as spherical image can be mapped onto each other by parallel normals. It is assumed, in addition, that principal directions at every point on S are mapped into principal directions at the image point on $\bar{S}$. Let $k_i(=1, 2)$ be the principal curvatures of $S, \bar{k}_i$ the corresponding principal curvatures of $\bar{S}$. Via the spherical image mapping, one may consider the function $\varphi = (k_1^{-1} - \bar{k}_1^{-1})\cdot(k_2^{-1}-\bar{k}_2^{-1})$ as being defined on the unit sphere Σ. We show: If φ does not change sign and appropriate boundary conditions are satisfied, then S differs from $\bar{S}$ by a translation. Since the spherical image mapping always preserves principal directions, one obtains in particular characterizations of the hemisphere. Further results for ovaloids $S, \bar{S}$ within this class of mappings: If $\bar{k}_1 \geqq k_1, \bar{k}_2 \geqq k_2$ everywhere, then a translate of $\bar{S}$ fits inside S; if S and $\bar{S}$ have the same total mean curvature, then $\int_\Sigma \varphi d\omega \leqq 0$ with equality if and only if S is a translate of $\bar{S}$.

Let $S, \bar{S}$ be smooth, oriented surfaces with positive Gaussian curvature in fixed position in E^3. Assume that their spherical images are simple and coincident, so that they can be mapped diffeomorphically onto each other by equal normals. We impose the additional condition that, under this standard mapping—which we shall henceforth call the *normal mapping*—, principal directions are preserved, i.e., that at every point on S there exists a pair of principal directions which are mapped into principal directions at the image point on $\bar{S}$. The normal mapping between surfaces of revolution with parallel axes of rotation has this property. If S is an ovaloid and $\bar{S}$ a sphere, then the normal mapping again certainly preserves principal directions. Further "trivial" examples are furnished by pairs of homothetic surfaces or pairs of surfaces which are parallel in the classical sense of Steiner. This last class has been investigated in [9]. Here various geometric conclusions will be drawn from the existence of such a mapping in the large in conjunction with given boundary conditions and inequalities connecting the principal curvatures of $S, \bar{S}$ at corresponding points. In particular, we shall obtain congruence theorems, characterizations of the sphere and statements about relative size. These geometric results are in part generalizations of our geometric results in [8]. They are for the most part direct consequences of the