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A NULLSTELLENSATZ FOR NASH RINGS

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Let D be a domain in R^n defined by a finite number of strict polynomial inequalities. Then the Nash ring A_D is the ring of real valued algebraic analytic functions defined on D. In this paper, it is shown that A_D is Noetherian and has a nullstellensatz. For $\mathscr{P}$ a prime ideal of A_D, $A_D/\mathscr{P}$ is said to be rank one orderable if its quotient field can be ordered over R so that it has essentially one infinitesimal. Then $A_D/\mathscr{P}$ is rank one orderable if and only if $\mathscr{P}$ equals the set of functions in A_D which vanish on the zero set of $\mathscr{P}$ in D.

DEFINITION 0.1. Let R denote the real numbers. Let D be a domain in R^n, defined by a finite number of polynomial inequalities $p_i(x) > 0$. A function $f: D \to R$ is said to be algebraic analytic if there exists a non-trivial polynomial $p_f(z, x_1, \cdots, x_n)$ in $R[z, x_1, \cdots, x_n]$ so that $p_f(f(x), x) = 0$ for all x in D, and if f is analytic (expandable in convergent power series) at every point of D.

DEFINITION 0.2. The ring of all such algebraic analytic functions $f: D \to R$ is called the Nash ring A_D; see [7] for this notation.

DEFINITION 0.3. (1) An ideal J of A_D is real if $\sum_{i=1}^{m} \lambda_i^2 \in J$ implies all $\lambda_i \in J$.

(2) For $J \subset A_D$, $V_R(J) = \{a \in R^n \mid f(a) = 0$ for all f in $J\}$.

(3) For $S \subset D$, $I(S) = \{f \in A_D \mid f(s) = 0$ for all s in $S\}$.

In § 1 and § 2 we develop some of the preliminaries for the study of the Nash ring. Most of § 1 comes form Cohen's paper [3]. In § 2 we prove the finiteness of the number of components of an algebraic set using Cohen's theory. In § 3 it is shown that A_D is Noetherian. Mike Artin made several valuable suggestions which were very helpful in proving this theorem.

Finally in § 4 we get to the nullstellensatz. Originally it was intended to prove the following conjecture.

CONJECTURE 0.4.[1] An ideal $J \subset A_D$ is real if and only if $I(V_R(J)) = J$.

Instead of this we are only able to show that: If $\mathscr{P} \subset A_D$ is prime, then $A_D/\mathscr{P}$ is rank one orderable (Definition 4.2) if and only if $I(V_R(\mathscr{P})) = \mathscr{P}$. This is sufficient to prove the conjecture in the case $D \subset R^2$. This is because the only nontrivial case is for $\mathscr{P}$ a prime of dimension 1 in which case $A_D/\mathscr{P}$ real implies $A_D/\mathscr{P}$ rank one orderable.

[1] Added in proof, this conjecture is now a theorem proved by T. Mostowski, preprint 1974.