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THE STRUCTURE OF GALOIS CONNECTIONS

ZAHAVA SHMUELY

This paper deals with Galois connections between two partially ordered sets (posets) A, B. The first sections are devoted to the construction of all Galois connections between A and B. The last sections deal with properties of $A \otimes B$, the set of mappings T: A $\rightarrow$ B which "participate" in a Galois connection between A and B, with the pointwise partial order.

Every Galois connection between two posets A, B can be uniquely extended to a Galois connection between $\nu(A)$ and $\nu(B)$, the completions by cuts of A, B resp., and $A \otimes B$ is characterized as a subset of $\nu(A) \otimes \nu(B)$. As an application we get: The completion by cuts of a residuated groupoid (semigroup) is a residuated groupoid (semigroup, resp.). The completion by cuts of a Brouwerian lattice is a Brouwerian lattice. The completion by cuts of a relation algebra is a relation algebra. When A and B are complete lattices, $A \otimes B$ is isomorphic to a certain set of semi-ideals of $A \times B$. This yields a procedure for constructing all Galois connections between any two posets. By dualization all sup-preserving and inf-preserving mappings are determined.

Bounded posets A, B are embedded in $A \otimes B$ in a peculiar way. $A \otimes B$ is a completely distributive, complete (Boolean) lattice iff A and B are completely distributive, complete (Boolean, resp.) lattices. Formal properties of $\otimes$ as a binary operation on bounded posets are investigated. In particular $A \otimes 2^B \cong A^B$ when A is a complete lattice, implying $A \otimes B^C \cong A^C \otimes B \cong (A \otimes B)^C$ when A, B are complete lattices and C is a poset. In certain respects, the behavior of $A \otimes B$ as a product of A and B resembles that of the tensor product of linear spaces.[1]

1. In the following, A, B, C denote partially ordered sets (posets). A^D is the dual A. A is *bounded* if it contains universal elements 0, 1 with $0 \leq p \leq 1$ for every $p \in A$. 2 is the poset $\{0, 1\}$ with $0 < 1$. A mapping $T: A \rightarrow B$ is *isotone* or *order-preserving* (*antitone*) whenever $p_1 \leq p_2$ in A implies $T(p_1) \leq T(p_2)$ $(T(p_1) \geq T(p_2))$. *Isomorphism* here means order-isomorphism. A is a *complete lattice* if every set $\{x_\alpha\} \subseteq A$ has a l.u.b., $\bigvee_\alpha x_\alpha$, and a g.l.b., $\bigwedge_\alpha x_\alpha$. A complete lattice is *completely distributive* whenever

$$\bigwedge_{\alpha\in\Omega}\bigvee_{\beta\in B_\alpha}x_{\alpha\beta} = \bigvee_{\phi\in\Pi B_\alpha}\bigwedge_{\alpha\in\Omega}x_{\alpha\phi(\alpha)}$$