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ON THE PRIME IDEAL DIVISORS OF $(a^n - b^n)$

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Let a and b denote nonzero elements of the ring of integers O_K of an algebraic number field K, such that ab^{-1} is not a root of unity and the principal ideals (a) and (b) are relatively prime.

DEFINITION 1. A prime ideal $\mathfrak{p}$ is called a *primitive prime divisor* of $(a^n - b^n)$ if $\mathfrak{p} \mid (a^n - b^n)$ and $\mathfrak{p} \nmid (a^k - b^k)$ for $k < n$.

DEFINITION 2. An integer n is called *exceptional for* $\{a, b\}$ if $(a^n - b^n)$ has no primitive prime divisors.

The set of integers exceptional for $\{a, b\}$ is denoted by $E(a, b)$. Using recent deep results of Baker, Schinzel [4] has proved that if $n > n_0(l)$ then $n \notin E(a, b)$, where $l = [K : Q]$ and n_0 is an effectively computable integer. In particular card $E(a, b) \leq n_0$. In this paper, using only elementary methods, upper bounds are obtained for card $\{n \in E(a, b) : n \leq x\}$ which are independent of a and b.

1. Introduction. The prime divisors of the sequence of rational integers $x_n = a^n - b^n$ have been studied by Birkhoff and Vandiver. They showed [**1**, p. 177] that if a and b are positive and relatively prime, then for $n > 6$ there is a prime p which divides $a^n - b^n$ and does not divide $a^k - b^k$ for $k < n$. Postnikova and Schinzel [**3**] have investigated analogues of this result for the ring of integers O_K of an algebraic number field K.

To fix our notation and terminology, a and b will always denote nonzero elements of O_K such that ab^{-1} is not a root of unity, and the principal ideals (a) and (b) are relatively prime. Note then that all the ideals $(a^n - b^n)$ are nonzero.

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DEFINITION 2. An integer n is called *exceptional for* $\{a, b\}$ if $(a^n - b^n)$ has no primitive prime divisors.

The set of integers exceptional for $\{a, b\}$ is denoted by $E(a, b)$. Using a theorem of Gelfond it can be shown [**3**, p. 172] that card $(E(a, b)) < n_0(a, b)$. Recently, using deep methods, Baker [**4**] has improved Gelfond's theorem, and has shown that card $E(a, b) < n_0(l)$, where $l = [K : Q]$. In this paper we obtain by elementary methods upper bounds for card $\{n \in E(a, b) : n \leq x\}$ which are independent of a and b. To state our theorem precisely we