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CLOSURE THEOREMS FOR AFFINE TRANSFORMATION GROUPS

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Let $\mathscr{H}$ be a closed subgroup of the group of linear transformations of R_n onto itself. Let hx denote the image of the point x under the transformation h, and let $\mathscr{G}$ be the transpose group of $\mathscr{H}$: i.e. its elements are associated with matrices which are the transposes of those in $\mathscr{H}$. For f in $L^2(R_n)$, let $Cl\{f; \mathscr{H} \times R_n\}$ denote the closure in the L^2 norm of the linear span of functions of the form $f(hx + t)$ where h is in $\mathscr{H}$, and t is in R_n. Since this space is translation-invariant, it is of the form $L^2(\hat{S})$: i.e. the set of L^2 functions $r(x)$ such that the nonzero set of $\hat{r}$, the Fourier transform of r, is, except for a set of measure zero, included in S. In the first theorem a precise description of S is given, and in the second, a function is constructed in a natural way whose translates alone generate the given space.

S is roughly the orbit of $N(f)$, the nonzero set of $\hat{f}$, under the group $\mathscr{G}$:

$$\bigcup_{g \text{ in } \mathscr{G}} g(N(f)) = \mathscr{G}(N(f)) .$$

However a difficulty arises in that $N(f)$ is determined only to within a set of measure zero, and $\mathscr{G}$ may transform sets of measure zero into nonmeasurable sets. For example, when the rotation group of the plane acts on a nonmeasurable linear set (of the x-axis, say), a nonmeasurable planar set results. Hence some care is required in defining S. Let E_f denote the set of points of density (one) of $N(f)$. Since the exceptional set of $N(f)$ has measure zero, every point of E_f has density one with respect to E_f. The orbit of the set E_f under the group $\mathscr{G}$ will be used as S, and as part of our first theorem, it will be shown that S is measurable.

The fact that closed translation-invariant subspaces of L^2 are of the form $L^2(\hat{S})$ is due to L. Schwartz [3]. The characterization of $Cl\{f; \mathscr{H} \times R_n\}$ in Theorem 1 reduces to the familiar Wiener theorem when $\mathscr{H}$ consists only of the identity and has been proved by S. R. Harasymiv for L^p spaces and for general distribution spaces [**1**, **2**] when $\mathscr{H}$ is the diagonal group. The second theorem involves the construction of a function p, arising naturally from f by an integration over $\mathscr{G}$, such that the translates of p generate the same space: i.e. such that $Cl\{p; R_n\} = Cl\{f; \mathscr{H} \times R_n\}$.