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THE ISOMETRIES OF $L^p(X,K)$

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Let (X, Σ, μ) be a finite measure space, and denote by $L^p(X, K)$ the Banach space of measurable functions F defined on X and taking values in a separable Hilbert space K, such that $\|F(x)\|^p$ is integrable. In this article a characterization is given of the linear isometries of $L^p(X, K)$ onto itself, for $1 \leqq p < \infty$, $p \neq 2$. It is shown that T is such an isometry iff T is of the form $(T(F))(x) = U(x)h(x)(\Phi(F))(x)$, where Φ is a set isomorphism of Σ onto itself, U is a weakly measurable operator-valued function such that $U(x)$ is a.e. an isometry of K onto itself, and h is a scalar function which is related to Φ via a formula involving Radon-Nikodym derivatives.

Throughout this paper the letter K will represent a separable Hilbert space which may be either real or complex. We denote by $\langle \cdot, \cdot \rangle$ the inner product in K, and by S the one-dimensional Hilbert space which is the scalar field associated with K.

A function F from X to K will be called measurable if the scalar function $\langle F, e \rangle$ is measurable for each $e \in K$. Then for $1 \leqq p < \infty$, we denote by $L^p(X, K)$ the Banach space of (equivalence classes of) measurable functions F from X to K for which the norm

$$\|F\|_p = \left\{\int \|F(x)\|^p d\mu\right\}^{1/p}, \qquad p < \infty ,$$
$$\|F\|_\infty = \text{ess sup} \|F(x)\|$$

is finite. (Here $\|\cdot\|_p$ denotes the norm in $L^p(X, K)$ and $L^p(X, S)$, and $\|\cdot\|$ that in K.) If $F \in L^p(X, K)$, we define the support of F to be the set $\{x \in X: F(x) \neq 0\}$.

Let $\{e_1, e_2, \cdots\}$ be some orthonormal basis for K. For $F \in L^p(X, K)$, we define the measurable coordinate functions f_n by $f_n(x) = \langle F(x), e_n \rangle$. Then almost everywhere we have $\sum_n |f_n(x)|^2 < \infty$, and $F(x) = \sum_n f_n(x)e_n$. Moreover, it is easily seen that each f_n belongs to $L^p(X, S)$.

Here we investigate the isometries of $L^p(X, K)$, for $1 \leqq p < \infty$, $p \neq 2$. For the case in which X is the unit interval, μ Lebesgue measure, and $K = S$, the isometries were determined by Banach in [1, p. 178]. In [4], Lamperti obtained a complete description of the isometries of $L^p(X, S)$ for an arbitrary finite measure space (X, Σ, μ).

Following Lamperti's terminology, we will call a mapping Φ of Σ onto itself, defined modulo null sets, a *regular set isomorphism* if it satisfies the properties