PACIFIC JOURNAL OF MATHEMATICS
Vol. 56, No. 2, 1975

A FUNCTION-THEORETIC APPROACH TO THE STUDY OF NONLINEAR RECURRING SEQUENCES

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For every real $r \geqq 0$, there is a sequence $\{b_n^{(r)}\}$ defined by

(1) $$b_0^{(r)} = 1, \quad b_{n+1}^{(r)} = \prod_{i=1}^{n} b_i^{(r)} + r \quad \text{for } n \geqq 0.$$

These sequences were considered previously, in [1], for integer values of r, and it was shown that there is a constant $\theta = \theta(r)$ such that

(2) $$b_{n+1}^{(r)} \sim \theta^{2^n}, \quad n \to \infty,$$

for each $r = 1, 2, 3, \cdots$. It was observed that

(3) $$b_{n+1}^{(2)} = 2^{2^n} + 1, \quad n \geqq 0,$$

whereby $\theta(2) = 2$, and the problem was proposed "to determine the algebraic or transcendental character of the real numbers $\theta(r)$ for $r = 1, 3, 4, 5, 6, \cdots$."

In this paper, we observe explicitly (in §II) that

(4) $$b_n^{(4)} = \tau^{2^n} + \tau^{-2^n} + 2, \quad n \geqq 1,$$

where $\tau = (\sqrt{5} + 1)/2 = 1.618 \cdots$ is the "Golden Mean", and thus $\theta(4) = \tau^2 = (\sqrt{5} + 3)/2 = 2.618 \cdots$.

Moreover, we extend the result (2) by showing, for every *real* $r > 0$, there is a real constant $\theta = \theta(r) > 1$ such that

(5) $$b_{n+1}^{(r)} = \theta^{2^n} + \frac{r}{2} + \frac{1}{8} r(r-2)\theta^{-2^n} + O(\theta^{-2^{n+2}}), \quad n \to \infty.$$

Thus for $r \neq 2$ the sequence $\{\beta_n^{(r)}\} = \{b_n^{(r)} - (r/2)\}$ differs from the sequence $\{\theta^{2^{n-1}}(r)\}$ by an amount which approaches 0 exponentially as $n \to \infty$. The case $r = 4$, described in (4), is illustrative of this behavior, while the case $r = 2$, described in (3), is exceptional in that the error term is identically 0.

For $r = 0$, $b_n^{(0)} = \beta_n^{(0)} = 1$ for all $n \geqq 0$, so that $\theta(0) = 1$; and $\theta(r)$ is a continuous, monotone increasing function of $r \geqq 0$.

The basic tool used in treating the general case is a new theorem in function theory (§III), which is ideally suited to the study of sequences