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BASICALLY BOUNDED FUNCTORS AND FLAT SHEAVES

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A functor is here called basically bounded if, roughly speaking, it is determined by its values on objects of some bounded cardinality. For functors on R-algebras, it is shown that common constructions involving basically bounded functors can again be computed on algebras of bounded size, and hence are uniquely defined irrespective of any special set-theoretic assumptions. Even operations which seem to require arbitrarily large algebras—computing Čech cohomology and sheafifications in the flat topology, forming Ext groups and sheaves—turn out to be basically bounded. The proofs use homological algebra and a notion of approximation by small coverings.

In abstract algebraic geometry, e.g. [4], it is often convenient to embed the category of schemes in a certain category of functors known as the sheaves for the flat (*fpqc*) topology. This larger category has the advantage of containing cokernels. But unfortunately their construction involves sheafification, which requires taking a direct limit over *all* flat coverings. Such a limit has no obvious reason to exist and indeed may well not exist; as we shall see, functors in general simply do not have flat sheafifications. One can rescue the construction *ad hoc* by restricting to a fixed "universe", but then the result will in general depend on the universe chosen. Yet all of this is unnatural; if we take an algebraic group acting on a variety over $\boldsymbol{Q}$, we never expect the quotient sheaf to have rational points whose existence depends on the size of the universe. It therefore ought to be possible to isolate the sheaves which have geometric meaning and deal with them in a way independent of arbitrary foundational assumptions. That is the purpose of this paper.

Much of the argumentation is category-theoretic. Indeed, the book of Gabriel and Ulmer [5], which came to my attention after this work was completed, overlaps part of the paper and provides a language for stating the results in extreme generality. In the interests of readability, however, I have chosen to present simply the detailed results for R-algebras and the flat topology. The basic special property of this topology is approximation by small coverings (§ 3), and with this in mind anyone who wants more general statements can easily abstract them.