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A GENERALIZED JENSEN'S INEQUALITY

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A generalized Jensen's inequality for conditional expectations of Bochner-integrable functions which extends the results of Dubins and Scalora is proved using a different method.

1. Introduction. Let $(\Omega, \mathbf{F}, P)$ be a probability space, $(\mathbf{U}, \|\cdot\|)$ a complex (or real) Banach space and $(\mathbf{V}, \|\cdot\|, \geqq_v)$ an ordered Banach space over the complex (or real) field such that the positive cone $\{v \in \mathbf{V} : v \geqq_v \theta\}$ is closed. Let x be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$. Let $\mathbf{G}$ be a sub-σ-field of the σ-field $\mathbf{F}$ and let f be a function on $\Omega \times \mathbf{U}$ to $\mathbf{V}$ such that for each $u \in \mathbf{U}$ the function $f(\cdot, u)$ is strongly measurable with respect to $\mathbf{G}$ and such that for each $\omega \in \Omega$ the function $f(\omega, \cdot)$ is continuous and convex in the sense that $tf(\omega, u_1) + (1-t)\, f(\omega, u_2) \geqq_v f(\omega, tu_1 + (1-t)u_2)$ whenever $u_1,\ u_2 \in \mathbf{U}$ and $0 \leqq t \leqq 1$. For any Bochner-integrable function z on $(\Omega, \mathbf{F}, P)$ to any Banach space $\mathbf{W}$, we define $E[z \mid \mathbf{G}]$ "a conditional expectation of z relative to $\mathbf{G}$" as a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{W}$ such that $E(z \mid \mathbf{G}]$ is strongly measurable with respect to $\mathbf{G}$ and that

$$\int_A E[z \mid \mathbf{G}](\omega) dP = \int_A z(\omega) dP, \qquad A \in \mathbf{G},$$

where the integrals are Bochner-integrals.

The purpose of this note is to prove the following generalized Jensen's inequality:

THEOREM. *If $f(\cdot, x(\cdot))$ is Bochner-integrable, then*

$$\text{(J)} \qquad E[f(\cdot, x(\cdot)) \mid \mathbf{G}](\omega) \geqq_v f(\omega, E[x \mid \mathbf{G}](\omega)) \qquad \text{a.e.}$$

The above theorem extends the results of Dubins **[2]** (cf. Mayer **[5**, p. 79]) and Scalora **[6**, p. 360, Theorem 2.3]. It is proved in **[2]** that the theorem is true for the case in which the spaces $\mathbf{U}$ and $\mathbf{V}$ are both the real numbers $\mathbf{R}$, while in **[6]** Scalora uses the methods of Hille-Phillips **[4]** to prove the theorem when the function $f(\omega, u)$ is replaced by a continuous, subadditive positive-homogeneous function $g(u)$ on $\mathbf{U}$ to $\mathbf{V}$. It should be noted that the method of the prôof used here is different than those used previously, the previous methods appear to be ineffective for a proof of the extension.