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RINGS WHOSE FAITHFUL LEFT IDEALS ARE COFAITHFUL

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A left module M over a ring R is cofaithful in case there is an embedding of R into a finite product of copies of M. Our main result states that a semiprime ring R is left Goldie, that is, has a semisimple Artinian left quotient ring, if and only if R satisfies (i) every faithful left ideal is cofaithful and (ii) every nonzero left ideal contains a nonzero uniform left ideal. The proof is elementary and does not make use of the Goldie and Lesieur-Croisot theorems. We show that (i) and (ii) are Morita invariant. Moreover, (ii) is invariant under polynomial extensions, and so is (i) for commutative rings. Absolutely torsion-free rings are studied.

The ring Q is a left classical quotient ring for the ring $R \subseteq Q$ if every regular element (nondivisor of zero) of R is invertible in Q and if every element of Q is of the form $b^{-1}a$ where $a, b \in R$ and b is regular; in this case we also say that R is a left order in Q. A ring is said to be left Goldie if it has the ascending chain condition on left annihilators and has finite uniform dimension. (A left R-module has finite uniform dimension if it has no infinite direct sum of nonzero submodules, and it is said to be uniform if it is nonzero and any two nonzero submodules have a nontrivial intersection.) A theorem of Goldie **[8, 9]** and Lesieur and Croisot **[12]** states that a ring is a left order in a semisimple Artinian ring if and only if it is semiprime and left Goldie. It is known that the ascending chain condition on left annihilators is not preserved under an equivalence of categories (Morita invariant); in fact, it does not go up to matrix rings. It is unknown whether being left Goldie is Morita invariant.

In section two we give a proof of the theorem stated in the abstract, and in the prime case we give a proof which shows directly that such a ring is an order in a full matrix ring over a division ring. We also weaken the hypothesis of an important theorem on semiprime PI rings. In the third section we use these techniques to study absolutely torsion-free rings. In particular, we show that an absolutely torsion-free ring is Goldie if and only if it has a uniform left ideal, and that the endomorphism ring of a finitely generated projective module over an absolutely torsion-free ring is absolutely torsion-free.