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TRANSITIVE AFFINE TRANSFORMATIONS ON GROUPS

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An affine transformation T on a group G is an automorphism followed by a translation; T is transitive if for each $x, y \in G$ there is an integer n such that $T^n(x) = y$. All groups with transitive affine transformations are determined: the infinite cyclic and infinite dihedral group are the only infinite examples; while the finite examples are semi-direct products of certain odd-order groups by a cyclic, dihedral or quaternion 2-group. The automorphism groups of the above groups are described, and the automorphisms which occur as parts of transitive affine transformations are given.

A bijective transformation T on a group G is called an *affine transformation* if there are an element s in G and an automorphism σ of G such that

$$(1) \qquad T(x) = s\sigma(x), \quad \text{for all} \quad x \in G.$$

We are interested in determining those groups G on which there is defined a *transitive* affine transformation T, i.e., for each pair x and y of elements in G there is an integer n such that $T^n(x) = y$. Groups having transitive affine transformations we shall call *single orbit groups*.

We shall first show, in §5, that there exist only two infinite single orbit groups; namely, the infinite cyclic group and the infinite dihedral group. The structure of all finite single orbit groups as semi-direct products is then described in §6, and presentations of these groups in terms of generators and relations are given in §7. We shall show that any such group is the semi-direct product of an odd-order group whose Sylow subgroups are cyclic by a 2-group which is cyclic, dihedral or a generalized quaternion group. Moreover, the image of the action of the 2-Sylow subgroup on the odd-order part must be a cyclic group. All groups of the above type have transitive affine transformations. In §8 the automorphism groups of the finite single orbit groups are calculated; it is shown that, with two types of exceptions, these automorphism groups can all be expressed in the same simple form.

If T is an affine transformation given as in (1), the element s will be called the *initial value* of T and σ the *associated automorphism*. Both are uniquely determined by T: $s = T(1)$ and $\sigma(x) = (T(1))^{-1}T(x)$ for all x in G. Given a single orbit group G, we shall determine in §9 all the associated automorphisms of transitive affine transformations on G.