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RADICALS OF SUPPLEMENTARY SEMILATTICE SUMS OF ASSOCIATIVE RINGS

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This paper deals with the effect of radicals (in the Kurosh-Amitsur sense) on supplementary semilattice sums of rings as defined by J. Weissglass (Proc. Amer. Math. Soc., 39 (1973), 471–473). It is shown that if $\mathscr{R}$ is a strict, hereditary radical class, then $\mathscr{R}(R) = \Sigma_{\alpha \in \Omega} \mathscr{R}(R_\alpha)$ for every supplementary semilattice sum $R = \Sigma_{\alpha \in \Omega} R_\alpha$ with finite Ω. If $\mathscr{R}$ is an A-radical class or the generalized nil radical class, the same conclusion holds with the finiteness restriction removed. On the other hand, if $\mathscr{R}(\Sigma_{\alpha \in \Omega} R_\alpha) = \Sigma_{\alpha \in \Omega} \mathscr{R}(R_\alpha)$ for all finite Ω, then $\mathscr{R}$ is strict and satisfies

(*) $R \in \mathscr{R} \Rightarrow$ the zeroring on the additive group of R belongs to $\mathscr{R}$,

a condition satisfied by both hereditary strict and A-radical classes.

Introduction. Semilattice sums of rings were introduced by Weissglass [11]. Let Ω be a *semilattice*, a commutative semigroup in which all elements are idempotent. A ring $R = \Sigma_{\alpha \in \Omega} R_\alpha$ is a *supplementary semilattice sum* of its subrings R_α if (i) $R^+ = \bigoplus_{\alpha \in \Omega} R_\alpha^+$ (here $(\ )^+$ denotes the additive group) i.e., R is a *supplementary sum* in the language of [3], and (ii) $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$. Examples include direct sums, polynomial rings and semigroup rings over semilattices.

In [11], Weissglass considered the inheritance of properties by a supplementary semilattice sum $R = \Sigma_{\alpha \in \Omega} R_\alpha$ from its subrings R_α. In [8], Janeski and Weissglass proved that R is regular if and only if each R_α is. Their arguments need minimal modification to obtain corresponding results in which regularity is replaced by various other hereditary radical properties, including quasi-regularity, nilness and local nilpotence.

We shall be concerned with a stronger condition on a radical class $\mathscr{R}$: $\mathscr{R}(\Sigma_{\alpha \in \Omega} R_\alpha) = \Sigma_{\alpha \in \Omega} \mathscr{R}(R_\alpha)$ (supplementary semilattice sum) for all (finite) supplementary semilattice sums $\Sigma_{\alpha \in \Omega} R_\alpha$.

For general information about radical classes the reader is referred to [3]. A radical class $\mathscr{R}$ is *strict* if every $\mathscr{R}$-subring S of a ring R is contained in $\mathscr{R}(R)$, or equivalently every subring of an $\mathscr{R}$-semi-simple ring is $\mathscr{R}$-semi-simple. See [9] for further details. An *A-radical* class [5] is one which contains with any ring R all ring S with $S^+ \cong R^+$. We denote the additive group of a ring by $(\ )^+$, the zeroring on an abelian group by $(\ )^0$; $<$ signifies an ideal. All rings considered are associative.