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COLLECTIVELY COMPACT SETS AND THE ERGODIC THEORY OF SEMI-GROUPS

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Let $\{T(t): t \geqq 0\}$ be a uniformly bounded semi-group of linear operators on a Banach space X such that 1 is an eigenvalue of each $T(t)$ and $T(a)$ is compact for some $a > 0$. Then the ergodic limit $A(t) = \lim_{n\to\infty} (1/n)\{T(t) + T^2(t) + \cdots + T^n(t)\}$ exists for each t. In this paper it is proved that if each $T(t)$, $t > 0$, is compact and 1 is, in a certain sense, an isolated eigenvalue of all $T(t)$, then for $t > 0$, the dimension of the null space of $T(t) - I$ is independent of t. Sufficient conditions are also obtained for the $\lim_{t\to\infty} T(t)$ to exist.

Suppose X is a real or complex Banach space. Let B denote the unit ball in X and $[X]$ the space of bounded linear operators on X into X. A set $\mathscr{K} \subset [X]$ is said to be collectively compact if $\mathscr{K}B = \{Kx: K \in \mathscr{K}, x \in B\}$ is relatively compact. Basic properties of such sets were obtained by Anselone and Palmer [1, 2]. Some of their results will be applied to semi-groups in the following sections.

2. Ergodic family associated with a semi-group. Let $\{T(t): t \geqq 0\}$ be a uniformly bounded semi-group of linear operators on a B-space X such that $T(a)$ is compact for some $a > 0$ and 1 is an eigenvalue of each $T(t)$. Then [4, VIII. 8.4] yields that

$$\begin{aligned}\lim_{n\to\infty} \cdot (1/n)\{T(t) + T^2(t) + \cdots + T^n(t)\} &\\ = \lim_{n\to\infty} \cdot (1/n)\{T(t) + \cdots + T(nt)\} &\\ = A(t) &\end{aligned}$$

say, exists in the uniform operator topology and defines a projection operator satisfying $A(t) = T(t)A(t) = A(t)T(t)$. Also, $A(t)$ is the residue operator in the Laurent expansion of $(\lambda - T(t))^{-1}$ in the neighbourhood of 1 and is represented by the Dunford integral

$$A(t) = (1/2\pi i)\int_C (\lambda - T(t))^{-1} d\lambda$$

where C is a sufficiently small circle with centre at 1 and will, in general, depend upon t. For $t = 0$, $T(0) = A(0) = I$ the identity operator. $\{A(t): t \geqq 0\}$ will be called the ergodic family associated with $\{T(t): t \geqq 0\}$.

When $T(a)$ is compact for some $a > 0$, the family $\{T(t): t \geqq a\}$ is collectively compact and totally bounded in the uniform operator