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HOMEOMORPHISMS OF THE PLANE

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This paper is concerned with homeomorphisms of Euclidean spaces onto themselves, with bounded orbits. The following results are obtained. (1) A homeomorphism of E^2 onto itself has both bounded orbits and an equicontinuous family of iterates iff it is a conjugate of either a rotation or a reflection; (2) An example of Bing is modified to produce a fixed point free, orientation preserving homeomorphism of E^3 onto itself, such that orbits of bounded sets are bounded; and (3) There is no homeomorphism of E^2 onto itself such that the orbit of every point is dense.

1. Introduction. One motivation for this paper is the well-known bounded orbit problem, "Does a homeomorphism T of E^2 onto itself, with bounded orbits, necessarily have a fixed point?" This is discussed in detail in § 2. In our investigations we were led to a study of homeomorphisms which have bounded orbits and an equicontinuous family of iterates, and we obtained a characterization of such homeomorphisms in Theorem 4. This theorem was proved earlier by Kerékjártó [13], using different methods. Our proof of this uses ε-sequential growths and is similar to the proof of the main theorem of [8].

In § 4, we study homeomorphisms with dense orbits.

2. The bounded orbit problem. As far as we know, this problem remains unsolved: Is there a homeomorphism T of the plane onto itself such that the orbit of each point is bounded, and which does not have a fixed point? The answer is "no" if T is orientation-preserving, and this is proved in [1, Proposition 1.2].

We wish to make the following observations:

(1) It follows from the methods of this paper that if there is a fixed point free homeomorphism T of the plane such that the orbits of bounded sets are bounded, then there is a compact continuum M in E^2, which does not separate the plane and which is invariant under T.

(2) If the orbits of points under T are bounded and closed, then T is periodic. This follows from [15].

(3) If T is orientation-reversing with bounded orbits, then T^2 is orientation-preserving with bounded orbits and thus T^2 has a fixed point. However, this does not necessarily imply that T has a fixed point. In [12], Johnson has given an example of a homeo-