PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 1, 1975

SOME PROPERTIES OF THE NASH BLOWING-UP

A. NOBILE

Intuitively, in the Nash blowing-up process each singular point of an algebraic (or analytic) variety is replaced by the limiting positions of tangent spaces (at non-singular points). The following properties of this process are shown: 1) It is, locally, a monoidal transform; 2) in characteristic zero, the process is trivial if and only if the variety is non-singular. Examples show that this is not true in characteristic $p > 0$; that, in general, the transform of a hypersurface is not locally a hypersurface; and that this process does not give, in general, minimal resolutions.

Introduction. In this paper, the term algebraic variety (over a field k) means reduced, separated algebraic scheme over k; the term analytic variety means reduced, separated analytic space over $\mathbf{C}$, the complex numbers. Let k be an algebraically closed field (resp. $k = \mathbf{C}$), X a reduced closed subscheme of a Zariski open $U \subset \mathbf{A}^n$ (resp. a reduced closed complex subspace of an open $U \subset \mathbf{C}^n$) of pure dimension r, defined by $\{f_1, \cdots, f_m\} \subset \Gamma(U, \mathcal{O}_U)$. By the Nash blowing-up of X we mean the pair (X^*, p) obtained by the following process. Let $S(X)$ be the set of singular points of X, X_0 its complement in X, $\eta: X_0 \rightarrow X \times G_r^n$ (G_r^n is the grassmanian of r-planes in n-space) the morphism determined by $\eta(x) = (x, T_{X,x})$ for each closed point $x \in X_0$ (here $T_{X,x}$ is the tangent space of X at x, which can be identified with an r-plane in n-space), X^* the closure of $\eta(X_0)$ in $X \times G_r^n$ (resp. the closure in the metric topology), $p: X^* \rightarrow X$ induced by the first projection. In the complex case it is not obvious that X^* is an analytic variety; see [7], Theorem 16.4 for a proof (or see Theorem 1 of this note).

It is possible to prove that (X^*, p) is (up to unique X-isomorphism) independent of the immersion (as a locally closed subset) of X in an affine space, hence the process globalizes.

Sketch of proof. Working (to simplify) in the algebraic case with closed points only, and calling $G_r(T) = \{r\text{-linear planes in } T\}$ for any vector space T, one verifies that $Z = \bigcup_{x \in X} x \times G_r(T_{X,x})$ is a subvariety of $X \times G_r^n$, and X^* is contained in Z. If X' is a locally closed in $\mathbf{A}^{m'}$, we have (using notations as above, but with primes): $X'^* \subset X' \times G_r^{m'}$. Assume $q: X \rightarrow X'$ is an isomorphism. Then,

$$(x, L) \rightarrow (q(x), dq(L)),$$