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A REPRESENTATION THEOREM FOR REAL CONVEX FUNCTIONS

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The Krein-Milman theorem is used to prove the following result. A nonnegative function f on [0, 1] is convex if, and only if, there exist nonnegative Borel measures μ_1 and μ_2 on [0, 1] such that

$$f(x)=\int_0^x(1-\xi)^{-1}(x-\xi)d\mu_1(\xi)+\int_x^1[1-(x/\xi)]d\mu_2(\xi),$$

for every $x \in [0,1]$. An example is given for which the representation is not unique.

1. Extremal elements. Let C be the set of all nonnegative convex real-valued functions on [0, 1]. Since the sum of two nonnegative convex functions is in C and since a nonnegative real multiple of a convex function is a convex function, the set C is a convex cone. It is the purpose of this paper to determine the extremal elements of this cone and to show that for the convex functions an integral representation in terms of extremal elements is possible (see [1] for terminology). We prove the following theorem which characterizes the extremal elements of C.

THEOREM 1. *The set of extremal elements of C consists of the following functions, where $m>0$:*

$e_+(m,\xi;x)=0$, $x\in[0,\xi]$ *and* $m(x-\xi)$ *for* $x\in[\xi,1]$, *where* $0\leqq\xi<1$;
$e_+(m,1;x)=0$, $x\in[0,1)$ *and* m *for* $x=1$;
$e_-(m,0;x)=0$, $x\in(0,1]$ *and* m *for* $x=0$;
$e_-(m,\xi;x)=m(\xi-x)$, $x\in[0,\xi]$ *and* 0 *for* $x\in[\xi,1]$ *where* $0<\xi\leqq 1$.

Proof. Let f be a function in C which assumes exactly one positive value in [0, 1]. If $f=c>0$, then $f(x)=cx+c(1-x)$ for $x\in[0,1]$ and hence, f is not an extremal element of C. If f is not constant, then f must be positive at one end point of [0, 1], since f is continuous on (0, 1) [5, p. 109]. It is evident that the two functions which are positive only at 0 and 1, respectively, are extremal elements of C. If $f\neq 0$ on [0, 1) and

$$f(1)\neq f_-(1)=\lim_{x\to 1-}f(x),$$