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A METRIC BASIS CHARACTERIZATION OF EUCLIDEAN SPACE

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In E^n there is exactly one line containing a given pair of points and exactly one k-flat containing a given k-simplex ($k+1$ points not contained in a lower dimensional space). The purpose of this paper is to prove converses of these propositions in the setting of complete, convex metric spaces. The most striking of these is given in Theorem 1 where it is proved that a complete, convex metric space which can be uniquely determined by any pair of points must be isometric with a subset of the real line. Theorem 2 is a higher dimensional analogue of this theorem. Metric characterizations of E^1 and E^n are derived from these results.

The distance between points p and q will be denoted by pq. A metric space M is convex if for each pair of distinct points p, q there exists a point r with $p \neq r \neq q$ and $pq = pr + rq$. M is externally convex if there exists a point s with $p \neq s \neq q$ and $sq = sp + pq$. For the remainder of this paper a space M will always mean a complete and convex metric space.

DEFINITION 1. A subset B of M is called a *metric basis* for M if $xz = yz$ for every z in B implies $x = y$.

It is easy to show that each pair of distinct points in E^n is a metric basis for the line it determines and that the vertices of a nondegenerate simplex form a metric basis for E^n.

The fact that each pair of points forms a metric basis for E^1 is almost characteristic of that space, as the following theorem shows.

THEOREM 1. *If each pair of distinct points of M forms a metric basis for M, then M is isometric with a subset of E^1.*

Proof. First we show that each three points of M are collinear. Suppose M contains points p, q, and r which are not collinear. By a theorem of Menger [5], each two points of M are the end points of a metric segment (isometric image of a line segment) in M. Let $pq \geqq pr$ and denote by $S(p, q)$ a fixed segment joining p and q. By the continuity of the metric and the fact that $S(p, q)$ is connected, it follows that there is a point x on $S(p, q)$ with $xq = xr$. If m is a