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CARATHÉODORY AND HELLY-NUMBERS OF CONVEX-PRODUCT-STRUCTURES

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Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be the Carathéodory-numbers of the convexity-structures $\mathscr{C}_1$ for X_1, respectivily $\mathscr{C}_2$ for X_2. It is shown that the Carathéodory-number $\mathfrak{c}$ of the convex-product-structure $\mathscr{C}_1 \oplus \mathscr{C}_2$ for $X_1 \times X_2$ satisfies the inequality $\mathfrak{c}_1 + \mathfrak{c}_2 - 2 \leqq \mathfrak{c} \leqq \mathfrak{c}_1 + \mathfrak{c}_2$; $\mathfrak{c}_1, \mathfrak{c}_2 \geqq 2$.

The upper bound for $\mathfrak{c}$ can be improved by one, resp. two, if a certain number, namely the so-called exchange-number, of one resp. each of the structures $\mathscr{C}_1$ and $\mathscr{C}_2$ is less than or equal to the Carathéodory-number of that structure.

A new definition of the Helly-number is given and Levi's theorem is proved with this new definition. Finally it is shown that the Helly-number of a convex-product-structure is the greater of the Helly-numbers of $\mathscr{C}_1$ and $\mathscr{C}_2$.

1. Preliminary remarks and definitions.

Existing notations and definitions have been taken from [3], [4] and, in particular, from [8]. Let $\mathscr{C}$ be a collection of subsets of a set X; by $\bigcap \mathscr{C}$ and $\bigcup \mathscr{C}$ we denote the intersection and the union respectivily, of the elements of $\mathscr{C}$. $\mathscr{C}$ is called a *convexity-structure* for X iff $\emptyset \in \mathscr{C}$, $X \in \mathscr{C}$ and $\bigcap \mathscr{F} \in \mathscr{C}$ for each subcollection $\mathscr{F} \subset \mathscr{C}$; the pair $(X, \mathscr{C})$ is called a *convexity-space*. The $\mathscr{C}$*-hull* of a set $S \subset X$, denoted by $\mathscr{C}(S)$, is defined by $\mathscr{C}(S) = \bigcap \{C \mid C \in \mathscr{C} \wedge S \subset C\}$. We shall write $\mathscr{C}(a_1, \cdots, a_n)$ instead of $\mathscr{C}(\{a_1, \cdots, a_n\})$, and $\mathscr{C}(p \cup (A \backslash a))$ instead of $\mathscr{C}(\{p\} \cup (A \backslash \{a\}))$.

Let X_i be a nonempty set and let $\mathscr{C}_i$ be a convexity-structure for X_i; $i = 1, 2$. Then $\mathscr{C}_1 \oplus \mathscr{C}_2 = \{A \times B \mid A \in \mathscr{C}_1 \wedge B \in \mathscr{C}_2\}$ is a convexity-structure for the Cartesian-product $X_1 \times X_2$. The pair $(X_1 \times X_2, \mathscr{C}_1 \oplus \mathscr{C}_2)$ is a called the *convex-product-space*, also called the *Eckhoff-space*. Note that the $\mathscr{C}_1 \oplus \mathscr{C}_2$-hull of $E \subset X_1 \times X_2$ is given by $(\mathscr{C}_1 \oplus \mathscr{C}_2)(E) = \mathscr{C}_1(\pi_1 E) \times \mathscr{C}_2(\pi_2 E)$, where π_1 is the projection of $X_1 \times X_2$ on X_i; $i = 1, 2$. Also note that if $e_1, e_2, e_3 \in X_1 \times X_2$ with $e_1 \neq e_2$ and $\pi_i(e_i) = \pi_i(e_3)$ for $i = 1, 2$, then $e_3 \in (\mathscr{C}_1 \oplus \mathscr{C}_2)(e_1, e_2)$.

2. The Carathéodory-number and the exchange-number.

A convexity-structure $\mathscr{C}$ for X is said to possess the *Carathéodory-number* $\mathfrak{c}$ if $\mathfrak{c}$ is the smallest nonnegative integer such that $\mathscr{C}(S) = \bigcup \{\mathscr{C}(T) \mid T \subset S \wedge |T| \leqq \mathfrak{c}\}$, for all $S \subset X$. The following lemma is an immediate consequence of this definition.