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RELATIONS BETWEEN PACKING AND COVERING NUMBERS OF A TREE

A. MEIR AND J. W. MOON

Let P_k denote the size of the largest subset of nodes of a tree T with n nodes such that the distance between any two nodes in the subset is at least $k+1$; let C_k denote the size of the smallest subset of nodes of T such that every node of T is at distance at most k from some node in the subset. We determine various relations involving P_k and C_k; in particular, we show that $P_k + kC_k \leqq n$ if $n \geqq k+1$ and that $P_{2k} = C_k$.

1. Introduction. The *distance* between nodes x and y in a graph G is the number $d(x, y)$ of edges in any shortest path in G that joins x and y. (For definitions not given here see [1] or [5].) A subset $\mathscr{P}$ of nodes of G is a *k-packing* if $d(x, y) > k$ for all pairs of distinct nodes x and y of $\mathscr{P}$; the *k-packing number* of G is the number $P_k = P_k(G)$ of nodes in any largest k-packing in G. A subset $\mathscr{C}$ of nodes of G is a *k-covering* if for every node x in G there is at least one node y in $\mathscr{C}$ such that $d(x, y) \leqq k$; the *k-covering number* of G is the number $C_k = C_k(G)$ of nodes in any smallest k-covering of G.

Our object here is to establish various relations between $P_k(T)$ and $C_k(T)$ when T is a tree with n nodes. We consider the case $k = 1$ in §2 and determine those values of α and β for which there exists a tree T such that $P_1(T) = \alpha$ and $C_1(T) = \beta$. We derive upper bounds for $P_k(T)$ and $C_k(T)$ in §3. In §4 we show that $P_k(T) + kC_k(T) \leqq n$ for any tree T with n nodes when $n \geqq k + 1$ and we show that this inequality is, in a sense, best possible. Finally, in §5 we show that $P_{2k} = C_k$.

The quantities $P_1(G)$ and $C_1(G)$ have been considered before under different names. For example, $P_1(G)$ and $C_1(G)$ are called the independence number and the domination number of G in [5; Chap. 13]; and they are called the coefficients of internal and external stability in [1; Chap. 4]. Some inequalities for $P_1(G)$ and $C_1(G)$ are given in [2; Chaps. 13 and 14] but some of these are unnecessarily weak when G is a tree.

2. Relations between P_1 and C_1. In what follows T will always denote an arbitrary tree with n nodes. For convenience, we shall frequently write P and C for $P_1(T)$ and $C_1(T)$.