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ENDOMORPHISM RINGS OF SELF-GENERATORS

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The group of R-homomorphisms $\mathrm{Hom}_R(M, A)$, where M, A are modules over a ring R, is, in a natural way, a module over the endomorphism ring S of M. Under certain weak assumptions on M, the following is true: $\mathrm{Hom}_R(M, -)$ carries injective envelopes of R-modules into injective envelopes of S-modules iff M generates all its submodules. Modules of the latter type are called self-generators. For M a self-generator, $\mathrm{Hom}_R(M, -)$ has additional properties concerning chain conditions and the socle. Many of the known results in this area, in particular those for M projective, are special cases of our main theorems.

Introduction. The question of how properties of a unitary right R-module $M = M_R$ are related to properties of its endomorphism ring S has been answered completely by the Morita theorems in case M is a progenerator. Then the functors $F = \mathrm{Hom}_R(M, -): \mathfrak{M}_R \rightarrow \mathfrak{M}_S$ and $H = M \otimes_R -: {}_R\mathfrak{M} \rightarrow {}_S\mathfrak{M}$ are equivalences and hence preserve and reflect all categorical properties of objects ($\mathfrak{M}_R$ denotes the category of unitary right R-modules).

Anderson [1] determined the finitely generated and projective modules M, for which H preserves injective envelopes and called them perfect injectors. Inspired by his paper, we investigate the analogous problem for F and introduce the notion of a "perfect coinjector" along the model of [1] (without restrictions on M). When R is a Dedekind domain, we have a structure theorem for perfect coinjectors (2.1). It yields a characterization of torsion modules flat over their endomorphism ring which generalizes that for $R = Z$ in [13, Th. 2]. In particular, the perfect coinjectors coincide with those modules generating all their submodules (self-generators) fors for the special choice of R. This is false for arbitrary R, but it is true (2.4) if certain assumptions, weaker than either "projective" or "generator", are made on M (e.g., $M = MT$ where T is the trace ideal of M).

Large classes of self-generators (§3) justify a closer look: The lattices of R-submodules of $A \in \mathfrak{M}_R$ and S-submodules of $\mathrm{Hom}(M, A)$ are intimately related, and so, as a consequence, are the chain conditions and Goldie dimension of A and $\mathrm{Hom}(M, A)$. These correspondences arise as a natural continuation of Sandomierski's results in [15]. Moreover, the self-generators $M = MT$ are exactly those modules, for which F preserves the properties "simple" and "essential" just as in the optimal case, i.e. M a vector space (resulting socle-formula: 4.5).