PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 2, 1975

A RADON NIKODYM THEOREM FOR WEIGHTS ON VON NEUMANN ALGEBRAS

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Let φ and ψ be normal positive linear functionals on a von Neumann algebra M such that $\psi \leqq \varphi$. Sakai proved the existence of a unique element $h \in M$ with $0 \leqq h \leqq 1$ such that $\psi(x) = \frac{1}{2}\varphi(hx + xh)$ for any $x \in M$. A generalization of this theorem is obtained for weights on von Neumann algebras. Let φ be a faithful normal semi-finite weight and ψ any weight on M majorized by φ. Then there is a unique element $h \in M$ with $0 \leqq h \leqq 1$ such that $\psi(x) = \frac{1}{2}\varphi(hx + xh)$ holds for x in a σ-weakly dense *-subalgebra of M. A stronger version is obtained when ψ is assumed to be a normal positive linear functional. Moreover counterexamples are given to show that in general one can not expect this relation to hold for every $x \in M^+$.

1. Introduction. Let M be a von Neumann algebra with a faithful normal state φ. Sakai proved that for any positive linear functional ψ on M such that $\psi \leqq \varphi$ there exists a unique element $h \in M$ such that $\psi(x) = \frac{1}{2}\varphi(hx + xh)$ for all $x \in M$ [6]. In [10] we established the relationship of this Radon Nikodym theorem with the Tomita-Takesaki theory for von Neumann algebras with a separating and cyclic vector. In fact in this paper we showed that from a slight generalization of Sakai's theorem, it follows that the resolvent $(\Delta - \omega)^{-1}$ of the modular operator Δ associated with a separating and cyclic vector ξ_0 for M, maps the set $M'\xi_0$ into $M\xi_0$ for any $\omega \in C$ with $|\omega| = 1$ and $\omega \neq 1$.

Combes has shown [2] that with every faithful normal semi-finite weight φ on a von Neumann algebra M is canonically associated a left Hilbert algebra. In this paper we use some of the techniques introduced in [9, 10] and the Tomita-Takesaki theory to obtain a generalization of Sakai's Radon Nikodym theorem for weights. If ψ is any weight majorized by φ we construct a Radon Nikodym derivative $h \in M$ with $0 \leqq h \leqq 1$. If $\mathscr{M}_\varphi$ denotes the subalgebra spanned by the set $\{x \in M^+, \varphi(x) < \infty\}$ we prove that $xh + hx \in \mathscr{M}_\varphi$ for any x in a certain σ-weakly dense *-subalgebra of M and that $\psi(x) = \frac{1}{2}\varphi(hx + xh)$. Moreover we give a counterexample to show that in general we can not expect that $xh + hx \in \mathscr{M}_\varphi$ for any $x \in \mathscr{M}_\varphi$ so that $\varphi(hx + xh)$ would not even be defined.

If ψ would be invariant with respect to the modular automor-