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THE DENSITY TOPOLOGY

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The *density topology* on the real line is a strengthening of the usual Euclidean topology which is intimately connected with the measure-theoretic structure of the reals. The purpose of this note is to treat the density topology from a modern topological viewpoint.

1. Introduction. A sprinkling of papers has been published by analysts on the density topology. We summarize the important results from the literature, and contribute some new ones. These mainly concern the characterization of certain subspaces, and consideration of cardinal invariants. Many of the topics we touch upon can be treated in more general measure-theoretic structures than the real line, but this does not appear to be particularly fruitful topologically. The organization of this paper is as follows: in §2, results stated explicitly or inherent in the literature are given; in §3, some new results are obtained, especially concerning various subspaces of X; in §4, the implications for X of various set-theoretic hypotheses are examined.

2. Definitions and results from the literature.

DEFINITION 2.1. A measurable set $E \subseteq \mathbf{R}$ has *density* d at x if

$$\lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

exists and equals d. Different authors use slightly different definitions of density but they all agree in case $d = 1$ and therefore determine the same topology. Denote by $\phi(E)$, $\{x \in \mathbf{R}\colon d(x, E) = 1\}$. Let $A \sim B$ mean $A \Delta B$ (the symmetric difference of A and B) is a nullset (i.e. has measure zero).

THEOREM 2.2. (See e.g. **[12]**.) *Let A be measurable. Then*
(1) $\phi(A) \sim A$,
(2) *if* $A \sim B$, *then* $\phi(A) = \phi(B)$,
(3) $\phi(\varnothing) = \varnothing$ *and* $\phi(\mathbf{R}) = \mathbf{R}$,
(4) $\phi(A \cap B) = \phi(A) \cap \phi(B)$,
(5) *if* $A \subseteq B$, *then* $\phi(A) \subseteq \phi(B)$.