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ON STARSHAPED SETS AND HELLY-TYPE THEOREMS

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Suppose an ordered pair of sets (S, K) in a linear topological space is of Helly type $(n+1, n)$, i.e., for every $n+1$ distinct points in S there is a point in K which sees at least n of them via S. Then if S is closed, K compact, and $n \geqq 3$, the nontrivial visibility sets in K are pairwise nondisjoint. Sufficient conditions are obtained for S to be starshaped.

Let S be a subset of a linear topological space L. For points x, y in S, we say x *sees* y *via* S if and only if the segment $[x, y]$ lies in S. Further, the set S is said to be *starshaped* if and only if there is some point p in S such that, for every x in S, p sees x via S.

If S and K are subsets of L, with every point x in S is associated its *visibility set* $K(x)$, the set of all points of K which x sees via S.

We shall say (S, K) *has Helly-type* (s, r), where r and s are positive integers, $r \leqq s$, if for every s distinct points in S there is a point on K seeing at least r of them via S. Clearly, if (S, K) has Helly-type (s, r), and $0 \leqq i \leqq r-1$, then (S, K) has Helly-type $(s-i, r-i)$.

In this paper we obtain a solution to a problem posed by Valentine, concerning sets of Helly type which are unions of a finite number of starshaped sets [**3**, Prob. 6.7, p. 178], and also obtain some related results. Breen [**1**] has given conditions in the plane for a simply connected set to be a union of two starshaped sets. We replace simple connectedness by the following:

For S and K subsets of a linear topological space L, we shall say the ordered pair (S, K) has the *triangle property* if the interior of every triangle having an edge on K and the other edges in S is itself a subset of S.

If S is a closed subset of a linear topological space L, K is a compact convex subset of L of dimension k and (S, K) has the triangle property, then $K(x)$ is compact and convex for each $x \in S$. If (S, K) is of Helly type (r, r), for $r \geqq k+1$, then by Helly's theorem $\cap \{K(x)\colon x \in S\} \neq \emptyset$, and S is starshaped. However, it is possible under certain conditions to weaken the hypothesis considerably, and yet reach the same conclusion.

A collection of sets $\mathcal{K}$ is said to have "*piercing number*" j or a j-*partition* for a positive integer j, if $\mathcal{K}$ can be represented as a union of j collections, each with a nonvoid intersection.

The classical result on j-partitions is a theorem by H. Hadwiger and H. DeBrunner [**2**], which for convenience we state here as Theorem 1.