PACIFIC JOURNAL OF MATHEMATICS
Vol 62, No 2, 1976

COMPLEX VECTOR FIELDS AND DIVISIBLE CHERN CLASSES

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This paper contains two theorems which relate the maximal number of independent sections of a complex bundle over a manifold to the Chern classes of the bundle and certain functional cohomology operations. The main theoretical result of the paper is a formula which relates the obstruction to a lifting in a fibration and a functional cohomology operation.

1. Introduction. Let M be a connected, closed, orientable, smooth manifold of dimension $2n$. If ω is a complex n-plane bundle over M, the complex span of ω is the maximal number of cross-sections of ω which are linearly independent over the complex numbers. In this paper, we consider the following question: when is complex span $\omega \geqq q$? Hopf's theorem says that complex span $\omega > 0$ if and only if ω has vanishing Euler class and the theorems of Thomas (**[10]** and **[11]**) give an effective answer in the case $q = 2$. We study this problem in the cases $q = 3, 4$ and establish theorems which give necessary and sufficient conditions for complex span $\omega \geqq q$ in terms of the Chern classes of ω and certain functional cohomology operations. The Chern class of ω in $H^{2i}(M; \mathbf{Z})$ is denoted by $c_i(\omega)$. If $\delta_p P^1$ denotes the Steenrod p-power P^1 followed by the Bockstein associated with reduction mod p, $\delta_p P^1_\omega(c(i - p + 1))$ denotes a subset of a functional operation defined on the universal Chern class $c(i - p + 1)$ and contained in $H^{2i}(M; \mathbf{Z})$. This subset will be described in the second section of this paper. If p is a prime, M is j-connected mod p if $H_i(M; \mathbf{Z}_p) = 0$, $1 \leqq i \leqq j$. In both theorems below, M is 1-connected mod 2 and 3.

THEOREM 1. *If n is even, $n \geqq 6$, then complex span $\omega \geqq 3$ if and only if $c_i(\omega) = 0$, $n - 2 \leqq i \leqq n$, and $0 \in \delta Sq^2_\omega(c(n-2))$.*

THEOREM 2. *If n is odd, $n \not\equiv 1 \pmod 3$, $n \geqq 9$, and M is 3-connected mod 2, then complex span $\omega \geqq 4$ if and only if $c_i(\omega) = 0$, $n - 3 \leqq i \leqq n$, $0 \in \delta Sq^2_\omega(c(n-3))$, and $0 \in \delta_3 P^1_\omega(c(n-3))$.*

In Theorem 1, if $n \not\equiv 0 \pmod 3$, and $n \equiv 2 \pmod 4$, the connectedness hypothesis can be dropped. Thomas and Gilmore **[11]** show that if M is 3-connected, then for every n, complex span $\omega \geqq 3$ if and only if $c_{n-2}(\omega) = 0$ and $c_n(\omega) = 0$. Gilmore **[2]** proves theorems similar to Theorems 1 and 2 in which he assumes that $H_2(M; \mathbf{Z})$ and $H_4(M; \mathbf{Z})$,