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SUBSPACES OF SYMMETRIC MATRICES CONTAINING MATRICES WITH A MULTIPLE FIRST EIGENVALUE

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Let $\mathcal{U}$ be an $(r-1)(2n-r+2)/2$ dimensional subspace of $n \times n$ real valued symmetric matrices. Then $\mathcal{U}$ contains a nonzero matrix whose greatest eigenvalue is at least of multiplicity r, if $2 \leqq r \leqq n-1$. This bound is best possible. We apply this result to prove the Bohnenblust generalization of Calabi's theorem. We extend these results to hermitian matrices.

1. Introduction.

Let $\mathcal{W}_n$ be the $n(n+1)/2$ dimensional vector space of all real valued $n \times n$ symmetric matrices. Let A belong to $\mathcal{W}_n$. Arrange the eigenvalues of A in decreasing order

$$\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots \geqq \lambda_n(A). \tag{1.1}$$

We say that $\lambda_1(A)$ is of multiplicity r if

$$\lambda_1(A) = \cdots = \lambda_r(A), \tag{1.2a}$$

$$\lambda_r(A) > \lambda_{r+1}(A). \tag{1.2b}$$

Let $\mathcal{U}$ be a subspace of $\mathcal{W}_n$ of dimension k. We consider the question of how large k has to be so that $\mathcal{U}$ must contain a nonzero matrix A which satisfies (1.2a) for a given r. The nontrivial case would be

$$2 \leqq r \leqq n-1. \tag{1.3}$$

Clearly for $r = n$ we must have $k = n(n+1)/2$ as $\mathcal{U}$ will contain the identity matrix I.

We now state our main result:

THEOREM 1. *Let $\mathcal{U}$ be a k dimensional subspace in the space $\mathcal{W}_n$ of $n \times n$ real valued matrices. Assume that an integer r satisfies the inequalities* (1.3).

If

$$k \geqq \kappa(r) \tag{1.4}$$

where