PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 1, 1976

SUBSTITUTION IN NASH FUNCTIONS

GUSTAVE EFROYMSON

Let D be a domain in R^n. In this paper D is assumed to be defined by a finite number of strict polynomial inequalities. A Nash function on D is a real valued analytic function $f(x)$ such that there exists a polynomial $p(z, x_1, \cdots, x_n)$ in $R[z, x_1, \cdots, x_n]$ such that $p(f(x), x) = 0$ for all x in D. Let A_D be the ring of such functions on D. For any real closed field L containing R, use the Tarski-Seidenberg theorem to extend f to a function from a domain D_L (defined by the same inequalities as D), $D_L \subseteqq L^{(n)}$, to L. Now let $\varphi\colon A_D \to L$ be a homomorphism. Since $R[x_1, \cdots, x_n] \subset A_D$, $\varphi x = (\varphi x_1, \cdots, \varphi x_n)$ is a well defined point in $L^{(n)}$ and is in D_L. So $f(\varphi x)$ is defined for any f in A_D. In this paper it is shown that $f(\varphi x) = \varphi f$. From this result one can deduce Mostowski's version of the Hilbert Nullstellensatz for A_D.

As for the Nullstellensatz, since D. Dubois [2], and J. J. Risler [8], independently proved the real Nullstellensatz for polynomial rings, there have been various successful attempts to extend the result to other types of rings, for example, [4], [9]. In [5], a partial result was obtained for Nash rings and then, in [7], T. Mostowski proved the Nullstellensatz for Nash rings. There is still a question as to whether the result holds for Nash rings on more general domains than those considered here.

1. Mostowski's theorem.

We first recall some definitions.

DEFINITION 1. A set C contained in R^n is said to be semi-algebraic if it is defined by Boolean operations (finite union, finite intersection, complement) on sets of the form $\{a \in R^n \mid p(a) > 0$, for $p(x)$ in $R[x_1, \cdots, x_n]\}$. That is, C is defined by a finite number of polynomial inequalities.

DEFINITION 2. Let D be a set defined by a finite intersection of sets of the form $\{a \in R^n \mid p(a) > 0\}$. Then $A_D = \{f\colon D \to R$ such that f is analytic on D and there exists a polynomial $p(z, x)$ in $R[z, x_1, \cdots, x_n]$ such that for all x in D, $p(f(x), x) = 0\}$. This ring is called the ring of Nash functions on D.

DEFINITION 3. We wish to define certain subrings of $A_D = A$. Namely, let $B_0 = R(x_1, \cdots, x_n) \cap A_D$. Let $B_1 = \bigvee B_0(\sqrt{f})$ for f in B_0 and $f > 0$ on D. Let $B_2 = \bigvee B_1(\sqrt{f})$ for f in B_1 and $f > 0$ on D.