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FUNDAMENTAL UNITS AND CYCLES IN THE PERIOD OF REAL QUADRATIC NUMBER FIELDS

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PART II

1. Fundamental unit in $Q(\sqrt{M})$ from the expansion of $\sqrt{M}$. In the first part of this paper we succeeded to state explicitly the periodic expansion of $\sqrt{M}$, M a squarefree natural number, for infinitely many classes $\sqrt{M}$, each containing infinite many numbers. There are 14 types of these infinitely many classes, and they will all be enumerated here for the calculation of the fundamental unit $e_f, |e_f|>1$, of the quadratic field $Q(\sqrt{M})$. There are many ways to calculate e_f. Many an elaborate mathematician like G. Degert [4] and H. Yokoi [7] have done so by finding the smallest solution of Pell's equation $x^2 - My^2 = 1$, or $x^2 - My^2 = \pm 4$, if the latter is solvable which necessitates $M \equiv 1 (\mathrm{mod}\, 4)$. Now to solve Pell's equation, poses another problem. If the expansion of $\sqrt{M}$ as a periodic continued fraction is known, the problem is solved. For numerical values of M, this causes arithmetic difficulties only. If M is just a symbol standing for any natural number, the challenge of stating the periodic expansion of $\sqrt{M}$ explicitly as a function of $\sqrt{M}$, has yet not been taken by mathematicians, except in a few cases enumerated by O. Perron [5]. These few cases have recently been enriched by a brilliant paper by Yamamoto [6], and by the author in [3]. Of course, $M = D^2 + d$, $1 \leqq d \leqq 2D$, and the author conjectures that if we know a functional relationship $D = D(d)$, the periodic expansion can be stated explicitly, as was indeed demonstrated by the author in the first part of this paper for certain arithmetic functions $D(d)$. But if the expansion of $\sqrt{M}$ as a periodic continued fraction is stated explicitly, the fundamental unit e_f of $Q(\sqrt{M})$ can be also stated explicitly by methods which are generally known, and will be briefly reviewed here. We shall also restate the notations and formulas of the first part of this paper of which we shall frequently make use here.

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(15.1)\quad \begin{cases} (\mathrm{i})\ \sqrt{M} = w = x = \dfrac{w + P_0}{Q_0} = b_0 + \dfrac{1}{x_1};\ P_0 = 0;\ Q_0 = 1\,; \\ \qquad b_0 = [w]\,; \\ (\mathrm{ii})\ x_k = \dfrac{w + P_k}{Q_k} = b_k + \dfrac{1}{x_{k+1}};\ P_k = b_{k-1}Q_{k-1} - P_{k-1}\,; \end{cases}
$$