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LATTICE ORDERINGS ON THE REAL FIELD

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Since every total order is a lattice order, and the real field R is a totally ordered field, it is a lattice-ordered field. In 1956 Birkhoff and Pierce raised the question of whether R can be made into a lattice-ordered field in any other way. In this paper we answer their question affirmatively by showing that there are, in fact, 2^c such orderings, where c is the cardinal of R.

Introduction. We answer the question of the existence of such orderings, raised by Birkhoff and Pierce in [2, p. 68], in Theorem 1, and find the number of orders in Corollary 1.2. We denote the rational field by $\boldsymbol{Q}$, the *positive cone* of $\boldsymbol{R}$ (i.e., the set of reals $\geqq 0$) in the usual order by $\boldsymbol{R}^+$, and the positive cone of $\boldsymbol{Q}$ by $\boldsymbol{Q}^+$.

THEOREM 1. *Let L be any subfield of $\boldsymbol{R}$ except $\boldsymbol{Q}$. Let K be any proper subfield of L, such that L is algebraic over K. Then there is a relation $\leqq$ on L, with positive cone P_L, such that $\langle L, \leqq \rangle$ is a lattice-ordered field which is not totally ordered. Moreover*:

(1) *The order $\leqq$ restricted to K is the usual total order* ($K \cap P_L = K \cap \boldsymbol{R}^+$).

(2) *K is the largest totally ordered subfield of L under $\leqq$.*

(3) *The order $\leqq$ is a distributive lattice order.*

(4) *The order $\leqq$ is **R-compatible*** ($P_L \subseteq \boldsymbol{R}^+$).

(5) *$L \cap \boldsymbol{R}^+$ is **quotient-represented** by P_L, in the sense that for each $l \in L \cap \boldsymbol{R}^+$, there exist $p, q \in P_L$ with $q \neq 0$, such that $l = p/q$.*

We will give the proof in Section 2, where we state the main lemma (see 2.2). We will use the assertion (2) in counting the number of such orders, and we will need the technical feature (5) in the construction process.

COROLLARY 1.1. *Let L be a subfield of $\boldsymbol{R}$ containing κ distinct subfields K such that L is algebraic over K. Then L admits at least κ distinct lattice orders.*

Proof. By (2), these distinct subfields give distinct orders.

COROLLARY 1.2. *$\boldsymbol{R}$ admits exactly 2^c and the algebraic numbers $\boldsymbol{A}$ admit exactly $2^{\aleph_0}$ lattice orders.*

Proof. $\boldsymbol{R}$ is known to be algebraic over 2^c distinct subfields and