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A RATIONAL OCTIC RECIPROCITY LAW

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A rational octic reciprocity theorem analogous to the rational biquadratic reciprocity theorem of Burde is proved.

Let p and q be distinct primes $\equiv 1 (\mathrm{mod}\ 4)$ such that $(p/q) = (q/p) = 1$. For such primes there are integers a, b, A, B with

$$(1)\qquad \begin{cases} p = a^2 + b^2,\ a \equiv 1 (\mathrm{mod}\ 2),\ b \equiv 0 (\mathrm{mod}\ 2)\ , \\ q = A^2 + B^2,\ A \equiv 1 (\mathrm{mod}\ 2),\ B \equiv 0 (\mathrm{mod}\ 2)\ . \end{cases}$$

Moreover it is well-known than $(A/q) = 1$, $(B/q) = (-1)^{(q-1)/4}$. If k is a quadratic residue $(\mathrm{mod}\ q)$ we set

$$\left(\frac{k}{q}\right)_4 = \begin{cases} +1, \text{ if } k \text{ is a biquadratic residue } (\mathrm{mod}\ q)\ , \\ -1, \text{ otherwise}\ . \end{cases}$$

In 1969 Burde [2] proved the following

THEOREM (Burde).

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = (-1)^{(q-1)/4} \left(\frac{aB - bA}{q}\right)\ .$$

Recently Brown [1] has posed the problem of finding an octic reciprocity law analogous to Burde's biquadratic law for distinct primes p and q with $p \equiv q \equiv 1 (\mathrm{mod}\ 8)$ and $(p/q)_4 = (q/p)_4 = 1$. It is the purpose of this paper to give such a law. From this point on we assume that p and q satisfy these conditions and set for any biquadratic residue $k (\mathrm{mod}\ q)$

$$\left(\frac{k}{q}\right)_8 = \begin{cases} +1, \text{ if } k \text{ is an octic residue } (\mathrm{mod}\ q)\ , \\ -1, \text{ otherwise}\ . \end{cases}$$

It is a familar result that there are integers c, d, C, D with

$$(2)\qquad \begin{cases} p = c^2 + 2d^2,\ c \equiv 1 (\mathrm{mod}\ 2),\ d \equiv 0 (\mathrm{mod}\ 2)\ , \\ q = C^2 + 2D^2,\ C \equiv 1 (\mathrm{mod}\ 2),\ D \equiv 0 (\mathrm{mod}\ 2)\ . \end{cases}$$

Moreover we have $(D/q) = 1$. Also from Burde's theorem we have

$$(3)\qquad \left(\frac{aB - bA}{q}\right) = 1\ ,$$

and from the law of biquadratic reciprocity after a little calculation we find that $(B/q)_4 = +1$. We prove