PACIFIC JOURNAL OF MATHEMATICS
Vol 64, No. 1, 1976

ROOTS OF THE EULER POLYNOMIALS

F. T. HOWARD

In this paper we prove some new theorems about the real and complex roots of the Euler polynomials. For each n we show how the real roots of $E_n(x)$ are distributed in the closed interval [1, 3]. We also show how the real roots of $E_n(x)$ are distributed in the arbitrary interval $[m, m+1]$ for n sufficiently large. Finally, we prove that if a and b are nonzero rational numbers and c is a square-free integer, then $E_n(x)$ has no roots of the form $a\sqrt{c}$, $c \neq 1$, or $a + b\sqrt{c}, c$ even, or $a + bi$, a and b integers.

1. Introduction.

The Euler polynomial $E_n(x)$ degree n can be defined as the unique polynomial satisfying

$$E_n(x+1) + E_n(x) = 2x^n \qquad (n \geqq 0). \tag{1.1}$$

These polynomials have been extensively studied; see [**3**, Chapter VI] and [**4**, Chapter II] for example. The first fifteen Euler polynomials are listed in [**5**, p. 477].

In this paper we are primarily concerned with the real roots of $E_n(x)$, though we also prove a few results about the complex roots. It is well known that if n is even, $n > 0$, then the only real roots of $E_n(x)$ in the closed interval [0, 1] are 0 and 1, while if n is odd the only real root in [0, 1] is 1/2. Brillhart [**1**] has pointed out that these are the only complex roots in the "critical strip" of all complex numbers $x + iy$, $0 \leqq x \leqq 1$. In the same paper Brillhart proved that $E_5(x)$ is the only Euler polynomial with a multiple root and that the Euler polynomials have no rational roots other than 0, 1, 1/2.

The main results in this paper are:

(1) On the closed interval [1, 3] we show how the real roots of $E_n(x)$ are distributed for each n.

(2) On each interval $[m, m+1]$, $m > 0$, we show how the real roots of $E_n(x)$ are distributed for n sufficiently large.

(3) Let a and b be nonzero rational numbers and let c and d be square-free integers. The polynomial $E_n(x)$ has no roots of the form $a\sqrt{c}$, $(c \neq 1)$, $a + b\sqrt{c}$ (c even), $a\sqrt{d} + b\sqrt{c}\, i$ (c and d of different parity); or $a + bi$ (a, b integers).

It is pointed out that results similar to (3) are also true for the Bernoulli polynomials.