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INFINITE GALOIS THEORY FOR COMMUTATIVE RINGS

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Let S be a commutative ring with identity. A group G of automorphisms of S is called locally finite, if for each $s \in S$, the set $\{\sigma(s): \sigma \in G\}$ is finite. Let R be the subring of G-invariant elements of S. An R-algebra T is called locally separable if every finite subset of T is contained in an R-separable subalgebra of T. For an R-separable subalgebra T of S and for G a locally finite group of automorphisms it is shown that T is the fixed ring for a group of automorphisms of S. If, in addition, it is assumed that S has finitely many idempotent elements, then it is shown that any locally separable subring T of S is the fixed ring for a locally finite group of automorphisms of S. Examples are included which show the scope of these theorems.

As in **[6]** the closure of G with respect to a G-stable subalgebra E of the Boolean algebra of all idempotent elements of S is the set of all automorphisms ρ of S for which there exist a positive integer n and idempotents $e_i \in E$ and automorphisms $\sigma_i \in G$, such that $\bigcup_{i=1}^{n} e_i = 1$ and $e_i \cdot \rho = e_i \cdot \sigma_i$ for $1 \leqq i \leqq n$. The closure of G with respect to the set of all idempotent elements of S will be called the Boolean closure of G.

1. Infinite Galois theory.

Throughout this section, G will be a locally finite group of automorphisms of a commutative ring S and R will be the subring of G-invariant elements of S. The following definition will be needed in §3.

DEFINITION. A ring S is called a Galois extension of a ring R with Galois group H if H is finite with $R = S^H$, and if there exist a positive integer n and elements x_i, y_i of S, $1 \leqq i \leqq n$, such that $\Sigma_{i=1}^{n} x_i \sigma(y_i) = \delta_{1,\sigma}$ for all $\sigma \in H$.

LEMMA 1.1. *Let G be a locally finite group of automorphisms of S with $R = S^G$. If T is an R-separable subalgebra of S and $H = \{\sigma \in G \mid \sigma|_T = 1_T\}$, then $[G: H] < \infty$.*

Proof. Let $\Sigma_{i=1}^{n} x_i \otimes y_i$ be a separability idempotent for T over R. Then $\Sigma_{i=1}^{n} x_i y_i = 1$, and, for every $t \in T$, $\Sigma_{i=1}^{n} t \cdot x_i \otimes y_i = \Sigma_{i=1}^{n} x_i \otimes y_i t$ in $T \otimes_R T$ **[4]**. Let $K = \{\sigma \in G: \sigma(y_i) = y_i,\ 1 \leqq i \leqq n\}$. Then $H \subseteq K$. But if $\sigma \in K$ and $t \in T$, then