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THE C^*-ALGEBRAS OF SOME REAL AND p-ADIC SOLVABLE GROUPS

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When G is a locally compact group, the unitary representation theory of G is the "same" as the *-representation theory of the group C^*-algebra $C^*(G)$. Hence it is of interest to determine the isomorphism class of $C^*(G)$ for a wide variety of groups G. Using methods suggested by papers of Z'ep and Delaroche, we determine explicitly the C^*-algebras of the "$ax+b$" groups over all nondiscrete locally compact fields and of a number of two-step solvable Lie groups. Only finitely many C^*-algebras arise as the group C^*-algebras of 3-dimensional simply connected Lie groups, and we characterize many of them. We also discuss the C^*-algebras of unipotent $\mathfrak{p}$-adic groups.

1. Introduction. The C^*-algebra of a locally compact group is easily defined as the enveloping C^*-algebra of the group L^1-algebra [11, 13.9.1], but until recently the only groups the structure of whose C^*-algebra was explicitly "known" have been abelian and compact groups and a few semi-simple Lie groups: $SL(2, \boldsymbol{C})$ [13], [10]; $SL(2, \boldsymbol{R})$ [19]; the other groups with the same universal covering group as $SL(2, \boldsymbol{R})$ [18]; and $Spin(4, 1)$ [5]. A fair amount is known about the C^*-algebras of nilpotent Lie groups (see, for instance, [21]), but the problem of characterizing the C^*-algebra of the Heisenberg group up to isomorphism among the family of all C^*-algebras having the same spectrum has proved difficult and remains unsolved. It is therefore interesting that Z'ep [23] has now noticed that the study of certain C^*-algebra extensions by Brown, Douglas and Fillmore [8] can be applied to characterize the C^*-algebra of the "improper $ax+b$ group," the affine group of the real line.

We extend Z'ep's method to study the C^*-algebras of other groups with "relatively few" infinite-dimensional irreducible representations. The affine groups of the affine lines over all nondiscrete locally compact fields K (the remaining important cases being $K=\boldsymbol{C}$ and K a $\mathfrak{p}$-adic field) are treated in §§ 2 and 3; the calculations are routine and the results are quite similar to Z'ep's. More interesting is the fact that similar methods can be applied to some groups with perhaps infinitely many inequivalent infinite-dimensional irreducible representations. A class of such groups, all of which are two-step solvable connected Lie groups, is studied in § 4. The simplest group in this class is the "proper $ax+b$ group," the connected component of the identity element in the group considered by Z'ep.