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AMALGAMATED SUMS OF ABELIAN l-GROUPS

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A class $\mathscr{K}$ of algebraic structures is said to have the amalgamation property if, whenever G, H_1, and H_2 are in $\mathscr{K}$ and $\sigma_1: G \to H_1$ and $\sigma_2: G \to H_2$ are embeddings, then for some L in $\mathscr{K}$ there are embeddings $\tau_1: H_1 \to L$ and $\tau_2: H_2 \to L$ such that $\sigma_1\tau_1 = \sigma_2\tau_2$. Since this property has important universal-algebraic implications, this author has attempted to determine which well-known classes of abelian lattice-ordered groups (l-groups) have the amalgamation property. Theorem 1 lists those that do, and Theorem 2 lists those that do not. Finally, we focus our attention on one important class — Archimedean l-groups — in which the amalgamation property fails, and derive some sufficient conditions on G, H_1, and H_2 for amalgamation to occur.

Unless otherwise stated, all l-groups are abelian. For the basic theory of l-groups, see [3]. We write $A \oplus^* B$ for the sum, lexicographically ordered from the right, of an l-group A and an o-group B, while we write $A \oplus B$, $\prod_i A_i$, $\Sigma_i A_i$ for the cardinal sum or product of l-groups, ordered componentwise. For the o-groups of reals and integers we reserve the letters R and Z. $\mathscr{C}(G)$ and $\mathscr{P}(G)$ denote respectively the poset of convex l-subgroups and the complete Boolean algebra of polar subgroups of G. If $S \subseteq G$ then $G(S)$ denotes the convex l-subgroup of G generated by S.

Referring to the definition in the first paragraph, we call $(G, H_1, H_2, \sigma_1, \sigma_2)$ an *amalgam* and say that τ_1 and τ_2 *embed* the amalgam in L. We shall occasionally simplify the notation by assuming that σ_1 and σ_2 are inclusion maps.

THEOREM 1. *The following classes have the amalgamation property*:

(a) *all* (abelian) *l-groups*

(b) *o-groups*

(c) *l-groups with a finite basis*

(d) *l-groups with ACC on $\mathscr{C}(G)$*

(e) *l-groups with DCC on $\mathscr{C}(G)$*

(f) *l-groups with ACC and DCC on $\mathscr{C}(G)$.*

(g) *direct sums of subgroups of R, that is, Archimedean l-groups with property* (F).

A universal-algebraic proof of (a) and (b) can be found in [6], and a constructive proof of (b), via Hahn embeddings, is found in