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ABSOLUTE SUMMABILITY OF WALSH-FOURIER SERIES

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We prove that for all $f \in \mathscr{H}^1$, $\sum_{k=1}^{\infty} (1/k)\,|\hat{f}(k)| \le K\,||f||_{\mathscr{H}^1}$, where $\mathscr{H}^1$ is the Walsh function analogue of the classical Hardy-space and $\hat{f}(k)$ is the k^{th} Walsh-Fourier coefficient of f. We obtain this as a consequence of its dual result: given a sequence $\{a_k\}$ of numbers such that $a_k = O(1/k)$, there exists a function $h \in$ BMO with $\hat{h}(k) = a_k$.

We study the relation between our results and the theory of differentiation on the Walsh group, developed by Butzer and Wagner.

Introduction. We are interested in various properties of Walsh-Fourier series. $w_k(\cdot)$ will denote the k^{th} Walsh function in the Paley-enumeration and $\hat{f}(k)$ will be the corresponding Walsh-Fourier coefficient of $f \in L^1$. $\mathscr{H}^1$ and BMO will denote the Walsh function analogues of the classical Hardy space and the functions of bounded mean oscillation, respectively (see [3], pp. 3-4; also refer to the section on "Preliminaries", in this paper).

Our principal result is

THEOREM 1. *There exists a constant $K > 0$ such that*

$$\sum_{k=1}^{\infty} (1/k)\,|\hat{f}(k)| \leqq K\,||f||_{\mathscr{H}^1}\,,$$

for all $f \in \mathscr{H}^1$.

Our proof of Theorem 1 does not follow the lines of its trigonometric analogue (see [5], pp. 286-287). We use the fact that Theorem 1 is equivalent to

THEOREM 2. *Given a sequence $\{a_k\}$ of numbers such that $a_k = O(1/k)$, there exists a function h in* BMO *such that $\hat{h}(k) = a_k$ for all k.*

We give a direct proof of Theorem 2.

Theorem 2, combined with a result of Fine [2] gives Lip $(1, L^1) \subsetneqq$ BMO. However, Lip $(1, L^1) \not\subset L^\infty$, in contrast with the trigonometric case where Lip $(1, L^1) = BV \subsetneqq L^\infty$ (see [5], p. 180). Theorem 2 also has connections with the Butzer-Wagner theory of differentiation on the Walsh-group (see [1]). The antidifferentiation kernel $W(x) \sim 1 + \sum_{k=1}^{\infty} (1/k) w_k(x)$ was shown by Butzer-Wagner to be in Lip $(1, L^1)$.