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CHAINED RINGS

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All rings considered are commutative with identity. A chained ring is any ring whose set of ideals is totally ordered by inclusion. The main object of this paper is stating conditions in which every valuation overring of a given ring is a chained ring. It is shown that every valuation overring of a ring R is a chained ring if and only if the ideal of zero divisors of $T(R)$, the total quotient ring of R, is the conductor of $\bar{R}$, the integral closure of R, in $T(R)$. An example is provided of a valuation ring which is not a chained ring even though its total quotient ring is a chained ring.

1. Preliminaries. In this short section we set notation and establish some elementary results.

A *total quotient ring* is a ring consisting entirely of zero divisors and units. If R is a ring, then let $T(R)$ denote the smallest total quotient ring containing R. We will call $T(R)$ *the total quotient ring of* R.

Let G be a totally ordered abelian group. A *valuation* on a ring T is a function v from T onto $G \cup \{\infty\}$, such that for all x and y in T:

(1) $v(xy) = v(x) + v(y)$;

(2) $v(x + y) \geqq \min\{v(x), v(y)\}$;

(3) $v(1) = 0$ and $v(0) = \infty$.

Let V be a ring, let P be a prime ideal of V, and let T be a ring containing V. Manis [11] defined the pair (V, P) to be a *valuation pair* of T if the following equivalent conditions are satisfied:

(a) If S is a subring of T containing V and if M is a prime ideal of S such that $M \cap V = P$, then $V = S$;

(b) For each x in $T \backslash V$, there exists y in P such that xy is in $V \backslash P$;

(c) There is a valuation (v, G) on T such that

$$V = \{x \in T \mid v(x) \geqq 0\} \quad \text{and} \quad P = \{x \in T \mid v(x) > 0\} .$$

If (V, P) is a valuation pair of T, then V is called a *valuation ring*. We will sometimes represent V by T_v. In this paper we always assume that T is $T(V)$. Chained rings are valuation rings, and in the domain case the concepts are equivalent to the concept of a valuation domain.

When we deal with rings containing zero divisors, we say that