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COMMUTATORS AND NUMERICAL RANGES OF POWERS OF OPERATORS

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If 0 does not lie in the closure of the numerical range of any positive integral power of a Hilbert space operator T, then an odd power of T is normal. If, in addition, T is convexoid, then T itself is normal; in fact, T is the direct sum of at most three rotated positive operators. A version of these results is given in terms of commutators.

1. Introduction. In [8] C. R. Johnson proved: For an $m \times m$ complex matrix A, if A^n is not normal for any positive integer n, then there exist a positive integer n_0 and a nonzero vector $x \in \boldsymbol{C}^m$ such that $(A^{n_0}x, x) = 0$. Later he and M. Neuman [9] obtained a number theoretic result which strengthens the above theorem. We generalize these theorems to the Hilbert space operator case in this paper.

Let $\mathscr{B}(\mathscr{H})$ denote the set of bounded operators on a Hilbert space $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, $\overline{W}(T)$ denotes the closure of the numerical range of T. Our main results are: If $0 \notin \overline{W}(T^n)$, $n = 1, 2, 3, \cdots$, then an odd power of T is normal; in fact, T is similiar to the direct sum of at most three rotated positive operators. Moreover, under the above hypothesis, T is normal if and only if T is convexoid.

These results can be applied to the theory of commutators: Let $\mathfrak{H}$ denote a separable infinite dimensional Hilbert space. For $T \in \mathscr{B}(\mathfrak{H})$, if $T^n \notin \{SX - XS\colon S, X \in \mathscr{B}(\mathfrak{H}),\ S \text{ positive}\}$, $n = 1, 2, 3, \cdots$, then there are an odd integer k and a compact operator K such that $T^k + K$ is normal; furthermore, T is a compact perturbation of a normal operator if and only if the essential numerical range of T is a polygon (possibly degenerate).

2. Preliminaries. Let $\boldsymbol{C}$ denote the set of complex numbers and $\boldsymbol{R}^+$ the set of strictly positive numbers. For $\Omega \subset \boldsymbol{C}$, $\mathrm{Co}(\Omega)$ denotes its convex hull; $\Omega^n = \{z^n\colon z \in \Omega\}$, n a positive integer. We write $\Omega > r$, r a real number, if Ω is a real subset and each number in Ω is greater than r. Let $\alpha, \beta \in \boldsymbol{C}$ and $\varepsilon \in (0, 1]$, $\Theta(\alpha, \beta; \varepsilon)$ denotes the closed elliptical disc with eccentricity ε and foci at α and β,

$$\Theta(\alpha, \beta; \varepsilon) = \{z \in \boldsymbol{C}\colon |z - \alpha| + |z - \beta| \leqq |\alpha - \beta|/\varepsilon\} .$$