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NEGATIVE THEOREMS ON GENERALIZED CONVEX APPROXIMATION

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In this paper we show that there exist functions $f \in C[-1, +1]$ with all $(r+1)$-st order divided differences uniformly bounded away from zero for r fixed ($f[x_0, x_1, \cdots, x_{r+1}] \geqq \delta > 0$ for fixed δ and all sets $x_0 < \cdots < x_{r+1}$ in $[-1, +1]$), for which infinitely many of the polynomials of best approximation to f do not have nonnnegative $(r+1)$-st derivatives on $[-1, +1]$.

1. Introduction. In [6]-[10] there appear many examples of functions f in $C[a, b]$ with nonnegative $(r+1)$-st divided differences there for which infinitely many of the polynomials of best approximation to f fail to have nonnegative $(r+1)$st derivatives. None of these examples has the $(r+1)$st divided differences uniformly bounded away from zero. In [11] Roulier shows that if $f \in C^{2r+2}[-1, +1]$ and if $f^{(r+1)}(x) \geqq \delta > 0$ on $[-1, 1]$ then for n sufficiently large the polynomial of best approximation of degree less than or equal to n has a positive $(r+1)$st derivative on $[-1, +1]$.

On the other hand for the case $r = 0$ Roulier in [12] shows that first divided differences of f uniformly bounded away from zero is not sufficient to insure that for n sufficiently large the polynomial of best approximation to f is increasing on $[-1, 1]$.

In this paper we extend the results of [12] to the case when $r \geqq 0$. The proofs are similar to those in [12] but make use of higher order divided differences and their properties.

2. Notation and preliminary concepts. For $n = 0, 1, 2, \cdots$ define H_n to be the set of all algebraic polynomials of degree less than or equal to n. For $f \in C[a, b]$, let

$$\|f\| = \sup\{|f(x)|: a \leqq x \leqq b\}.$$

We define the degree of approximation to f to be

$$E_n(f) = \inf\{\|f - p\|: p \in H_n\},$$

$n = 0, 1, 2, \cdots$. It is well-known that there is a unique $p_n \in H_n$ for which $\|f - p_n\| = E_n(f)$. This p_n is called the *polynomial of best approximation to f on $[a, b]$ from H_n*. Unless specifically stated otherwise we will restrict ourselves to the interval $[-1, +1]$.

Define C^* *to be the class of continuous 2π-periodic functions* and H_n^* the trigonometric polynomials of degree n or less. Then