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A GEOMETRIC CHARACTERIZATION OF INDETERMINATE MOMENT SEQUENCES

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Hamburger and Stieltjes moment sequences are studied from the standpoint of the geometry of their moment spaces. Necessary and sufficient conditions are obtained that each of these sequences be indeterminate. The elements in the associated Jacobi and Stieltjes type continued fractions are characterized in terms of ratios of distances in the moment spaces.

1. Introduction. A sequence of real numbers $\{c_n\}_{n=0}^{\infty}$ is an H (Hamburger moment) sequence if there exists a bounded nondecreasing function γ on $(-\infty, \infty)$ such that

$$c_n = \int_{-\infty}^{\infty} t^n d\gamma(t) \qquad (n = 0, 1, 2, \cdots) . \tag{1}$$

The function γ, called a mass function for the sequence $\{c_n\}$, is normalized to be left continuous and such that $\gamma(0) = 0$. The sequence $\{c_n\}_{n=0}^{\infty}$ is an S (Stieltjes moment) sequence if it is an H sequence and there is a mass function γ for the sequence that is constant on $(-\infty, 0)$. An H sequence or an S sequence is determinate if the mass function γ for the sequence is unique. Otherwise the moment sequence is indeterminate.

The geometric approach of Carathéodory [2] for the classical moment problems has been extended and generalized by a number of authors (see [5]). In particular, Krein [6] initiated a geometric study of general Tchebycheff systems and Karlin and Shapley [4] rekindled interest in the geometry of moment sequences by their definitive memoir on the finite (Hausdorff) moment problem. The primary purpose of this paper is to provide, in the spirit of the works of Krein and of Karlin and Shapley, geometric characterizations for indeterminate H sequences and for indeterminate S sequences.

More specifically, let $\mathfrak{M}_{2m+1}$ denote the set of vectors $c = (c_0, c_1, \cdots, c_{2m})$ in Euclidean E^{2m+1} space such that there is a mass function γ on $(-\infty, \infty)$ for which (1) holds when $n = 0, 1, 2, \cdots, 2m$. For real $\lambda > 0$ and for c, c^* in $\mathfrak{M}_{2m+1}$, the vectors λc and $c + c^*$ are also in $\mathfrak{M}_{2m+1}$. Thus $\mathfrak{M}_{2m+1}$ is a convex cone in E^{2m+1}. For a given $c = (c_0, c_1, \cdots, c_{2m}) \in \mathfrak{M}_{2m+1}$, we consider the two dimensional