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THE RADICAL OF A REFLEXIVE OPERATOR ALGEBRA

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The radical of a reflexive operator algebra $\mathfrak{A}$ whose lattice of invariant subspaces $\mathfrak{L}$ is commutative is related to the space of lattice homomorphisms of $\mathfrak{L}$ onto $\{0, 1\}$. To each such homomorphism ϕ is associated a closed, two-sided ideal $\mathfrak{A}_\phi$ contained in $\mathfrak{A}$. The intersection of the $\mathfrak{A}_\phi$ is contained in the radical; it is conjectured that equality always holds. The conjecture is proven for a variety of special cases: countable direct sums of nest algebras; finite direct sums of algebras which satisfy the conjecture; algebras whose lattice of invariant subspaces is finite; algebras whose lattice of invariant subspaces is isomorphic to the lattice of nonincreasing sequences with values in $N \cup \{\infty\}$.

1. Introduction. This paper studies the radical of a certain class of non-self-adjoint operator algebras. Given an algebra $\mathfrak{A}$ and a lattice $\mathfrak{L}$ of orthogonal projections acting on a separable Hilbert space $\mathfrak{H}$, we use the standard notations, $\mathrm{Lat}\,\mathfrak{A}$ and $\mathrm{Alg}\,\mathfrak{L}$, to denote, respectively, the lattice of all projections invariant under $\mathfrak{A}$ and the algebra of all (bounded) operators which leave invariant each projection of $\mathfrak{L}$. $\mathfrak{A}$ and $\mathfrak{L}$ are said to be *reflexive* if $\mathfrak{A} = \mathrm{Alg}\,\mathrm{Lat}\,\mathfrak{A}$ and $\mathfrak{L} = \mathrm{Lat}\,\mathrm{Alg}\,\mathfrak{L}$, respectively. The algebras which we study are reflexive algebras which contain a maximal abelian self-adjoint algebra (m.a.s.a.).

A *commutative subspace lattice* is a lattice of pairwise commuting, orthogonal projections on $\mathfrak{H}$ which contains 0 and 1 and which is closed in the strong operator topology. It follows automatically that a commutative subspace lattice is a complete lattice. If $\mathfrak{A}$ is an operator algebra containing a m.a.s.a., then $\mathrm{Lat}\,\mathfrak{A}$ is a commutative subspace lattice. Every commutative subspace lattice, $\mathfrak{L}$, is reflexive ([1], p. 468), and $\mathrm{Alg}\,\mathfrak{L}$ is a reflexive algebra which contains a m.a.s.a. Henceforth, all lattices of projections in this paper will be commutative subspace lattices and all algebras will be reflexive algebras which contain a m.a.s.a. An incisive study of these lattices and algebras by Arveson is found in [1].

At least in certain special cases, the radical of a reflexive algebra, $\mathfrak{A}$, containing a m.a.s.a. can be described in terms of the set of lattice homomorphisms from $\mathfrak{L} = \mathrm{Lat}\,\mathfrak{A}$ onto $\{0, 1\}$. To each such homomorphism ϕ we shall associate a closed two-sided ideal $\mathfrak{A}_\phi$ in $\mathfrak{A}$. The radical, $\mathfrak{R}$, of $\mathfrak{A}$ is equal to the intersection of these ideals. It appears reasonable to conjecture that this equality holds for all